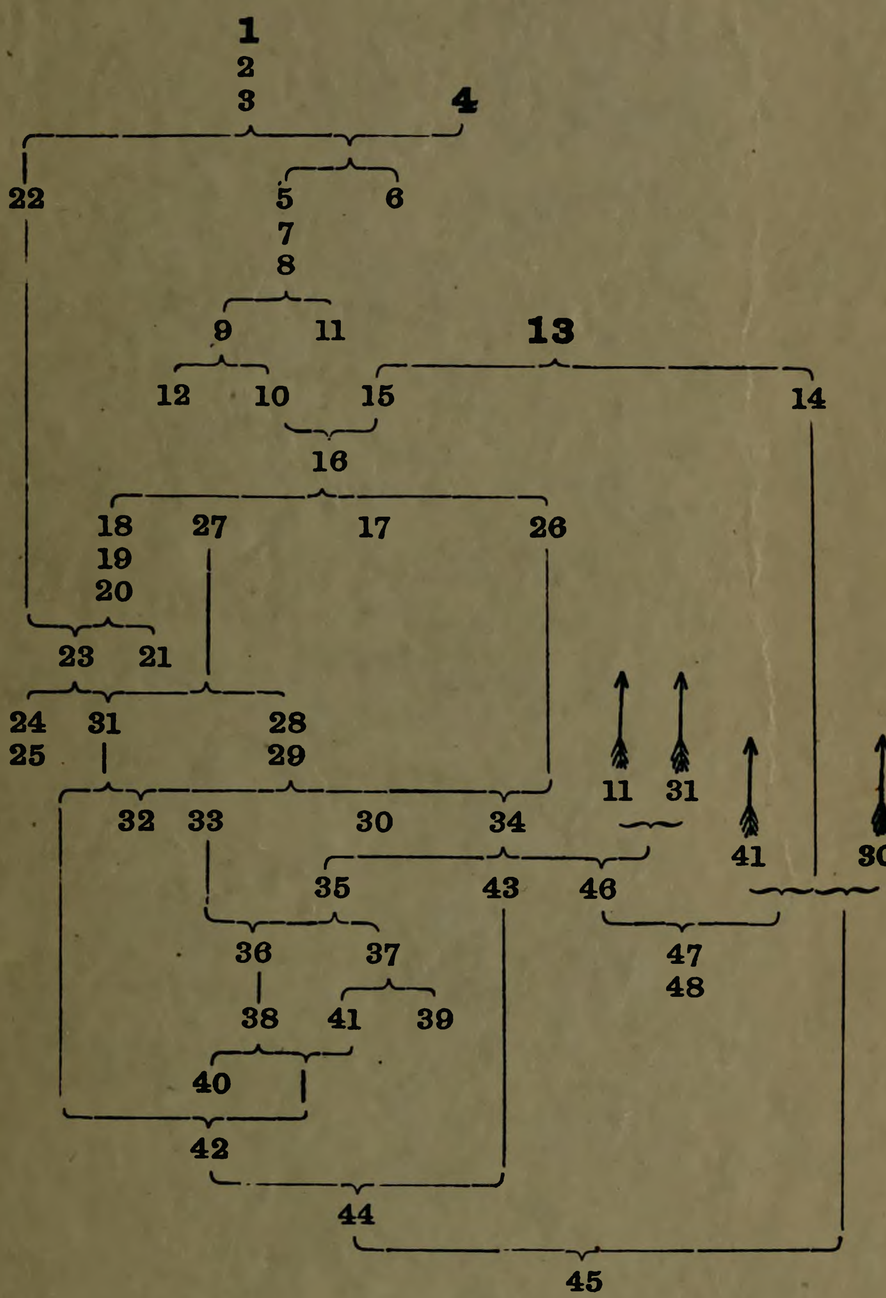
EUCLID, BOOK I.

Arranged in Logical Sequence.



EUCLID

BOOKS I, II.

C. L. DODGSON.

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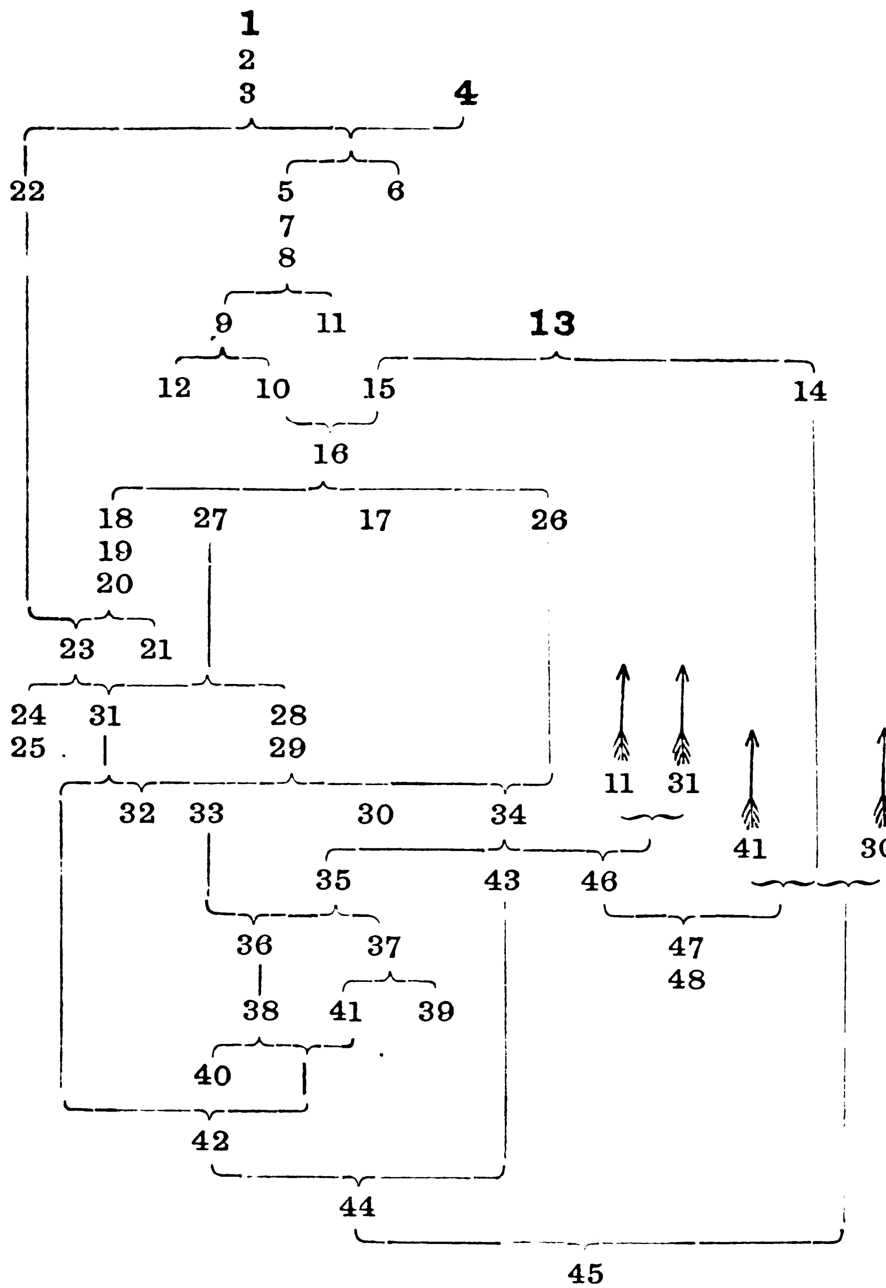

EUCLID

BOOKS I, II

EDITED BY

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INTRODUCTION.

PART I.

ADDRESSED TO THE TEACHER.

In preparing this edition of the first Two Books of Euclid, my aim has been to show what Euclid's method really is in itself, when stripped of all accidental verbiage and repetition. With this object, I have held myself free to alter and abridge the language wherever it seemed desirable, so long as I made no real change in his methods of proof, or in his logical sequence.

This logical sequence, which has been for so many centuries familiar to students of Geometry—so that 'The Forty-Seventh Proposition' is as clear a reference as if one were to quote the enuntiation in full—it has lately been proposed to supersede: partly from the instinctive passion for novelty which, even if Euclid's system were the best possible, would still desire a change; partly from the tacitly assumed theory that modern lights are necessarily better than ancient ones. I am not now speaking of writers who retain unaltered Euclid's sequence and numbering of propositions, and merely substitute new proofs, or interpolate new deductions, but of those who reject his system altogether, and, taking up the subject *de novo*, attempt to teach Geometry by methods of their own.

Some of these rival systems I have examined with much care (I may specify Chauvenet, Cooley, Cuthbertson, Henrici, Legendre, Loomis, Morell, Pierce, Reynolds, Willock, Wilson, Wright, and the Syllabus put forth by the Association for the

Improvement of Geometrical Teaching), and I feel deeply convinced that, for purposes of teaching, no treatise has yet appeared worthy to supersede that of Euclid.

It can never be too constantly, or too distinctly, stated that, for the purpose of teaching *beginners* the subject-matter of Euclid I, II, we do *not* need a complete collection of all known propositions (probably some thousands) which come within that limit, but simply a selection of some of the best of them, in a logically arranged sequence. In both these respects, I hold that Euclid's treatise is, at present, not only unequalled, but unapproached.

For the diagrams used in this book I am indebted to the great kindness of Mr. Todhunter, who has most generously allowed me to make use of the series prepared for his own edition of Euclid.

I will here enumerate, under the three headings of 'Additions' 'Omissions' and 'Alterations,' the chief points of difference between this and the ordinary editions of Euclid, and will state my reasons for adopting them.

1. Additions.

DEF. &c. § 11. The Axiom 'Two different right Lines cannot have a common segment' (in 3 equivalent forms). This is tacitly assumed by Euclid, all through the two Books (see Note to Prop. 4), and it is so distinctly analogous to his 'two right Lines cannot enclose a Superficies' that it seems desirable to have it formally stated.

DEF. &c. § 20. Here, to Euclid's Postulate 'A Circle can be described about any Centre, and at any distance from it,' I have added the words 'i.e. so that its Circumference shall pass through any given Point.' This I believe to be Euclid's real meaning. Modern critics have attempted to identify this given 'distance' with 'length of a given right Line,' and have then very plausibly pointed to Props. 2, 3, as an instance of unnecessary length of argument. 'Why does he not,' they say,

'solve Prop. 3 by simply drawing a Circle with radius equal to the given Line?' All this involves the tacit assumption that the 'distance' (*διάστημα*, i.e. 'interval' or 'difference of position') between two Points is equal to the length of the right Line joining them. Now it may be granted that this 'distance' is merely an abbreviation for the phrase 'length of the shortest path by which a Point can pass from one position to the other:' and also that this path is (as any path would be) a *Line:* but that it is a *right* Line is just what Euclid did *not* mean to assume: for this would make Prop. 20 an Axiom. Euclid contemplates the 'distance' between two Points as a magnitude that exists quite independently of any Line being drawn to join them (in Prop. 12 he talks of the 'distance CD' without joining the points C, D), and, as he has no means of *measuring* this distance, so neither has he any means of *transferring* it, as the critics would suggest. Hence Props. 2, 3, are logically necessary to prove the *possibility*, with the given Postulates, of cutting off a Line equal to a given Line. When once this has been proved, it can be done *practically* in any way that is most convenient.

AXIOMS, § 9. This is quite as axiomatic as the one tacitly assumed by Euclid (in Props. 7, 18, 21, 24), viz. 'If one magnitude be greater than a second, and the second greater than a third: the first is greater than the third.' Mine is shorter, and has also the advantage of saving a step in the argument: e.g. in Prop. 7, Euclid proves that the angle ADC is greater than the angle BCD, a fact that is of no use in itself, and is only needed as a step to another fact: this step I dispense with.

PROP. 8. Here I assert of all *three* angles what Euclid asserts of *one* only. But his Proposition *virtually* contains mine, as it may be proved three times over, with different sets of bases.

PROP. 24. Euclid contents himself with proving the first case, no doubt assuming that the reader can prove the rest for himself. The ordinary way of making the argument complete, viz. to interpolate 'of the two sides DE, DF, let DE be not greater than DF,' is very unsatisfactory: for, though it is *true* that, on this hypothesis, F will fall outside the Triangle

DEG, yet no *proof* of this is given. The Theorem, as here completed, is distinctly analogous to Prop. 7.

Book II, Def. § 4. The introduction of this one word 'projection' enables us to give, in Props. 12, 13, alternative enuntiations which will, I think, be found much more easy to grasp than the existing ones.

Book II, Prop. 8. Considering that this Proposition, with the ordinary proof, is now constantly omitted by Students, under the belief that Examiners never set it, I venture to suggest this shorter method of proving it, in hopes of recalling attention to a Theorem which, though not quoted in the Six Books of Euclid, is useful in Conic Sections.

(*Another proof of Euc. II.* 8.)

[Instead of 'On *AD* describe' &c, read as follows:—

A C B D

Square of *AD* is equal to
squares of *AB*, *BD*, with twice rectangle of *AB*, *BD*; [II. 4.
i. e. to squares of *AB*, *BC*, with twice rectangle of *AB*, *BC*;
i. e. to twice rectangle of *AB*, *BC*, with square of *AC*, [II. 7.
with twice rectangle of *AB*, *BC*;
i. e. to four times rectangle of *AB*, *BC*, with square of *AC*.
Q.E.D.]

2. Omissions.

Euclid gives separate Definitions for 'plane angle' and 'plane rectilineal angle.' I have ignored the existence of any angles other than rectilineal, as I see no reason for mentioning them in a book meant for beginners.

Prop. 11. Here I omit the Corollary (introduced by Simson) 'Two Lines cannot have a common segment,' for several reasons. First, it is not Euclid's: secondly, it is assumed as an Axiom, at least as early as Prop. 4: thirdly, the proof, offered for it, is illogical, since, in order to draw a Line from *B* at right angles to *AB*, we must produce *AB*; and as this can, *ex hypothesi*, be done in two different ways, we shall have two constructions, and therefore two perpendiculars to deal with.

PROP. 46. Here instead of drawing a Line, at right angles to *AB*, *longer* than it, and then cutting off a piece equal to it, I have combined the two processes into one, following the example which Euclid himself has set in Prop. 16.

3. Alterations.

DEFINITIONS, &c. § 7. Instead of the usual 'A straight Line is that which lies evenly between its extreme Points,' I have expressed it 'A right Line,' ('right' is more in harmony, than 'straight,' with the term 'rectilineal') 'is one that lies evenly as to Points in it.' This is Euclid's expression: and it is applicable (which the other is not) to infinite Lines.

PROP. 12. Here I bisect the angle *FCG* instead of the Line *FG*: i.e. I use Prop. 9 instead of Prop. 10. The usual construction really uses *both*, for Prop. 10 requires Prop. 9.

PROP. 16. Here, instead of saying that it 'may be proved, by bisecting *BC* &c. that the angle *BCG* is greater than the angle *ABC*,' I simply point out that it *has* been proved—on the principle that, when a Theorem has once been proved for *one* case, it may be taken as proved for all similar cases.

PROP. 30. Here Euclid has contented himself, as he often does, with proving *one* case only. But unfortunately the one he has chosen is the one that least needs proof: for, if it be given that neither of the outside Lines cuts the (infinitely producible) middle Line, it is obvious that they cannot meet each other.

BOOK II, PROP. 14. Here, instead of producing *DE* to *H*, I have drawn *EH* at right angles to *BF*. This at once supplies us with the fact that *GEH* is a right angle, without the necessity of tacitly assuming, as Euclid does, that 'if one of the two adjacent angles, which one Line makes with another, be right, so also is the other.'

PART II.

ADDRESSED TO THE STUDENT.

The student is recommended to read the Two Books in the following order, making sure that he has thoroughly mastered each Section before beginning the next.

BOOK I.

BOOK II.

BOOK I.

DEFINITIONS, &c.

[N.B. *Certain Postulates and Axioms are here inserted, to exhibit their logical connection with the Definitions: all these will be repeated, for convenience of reference, under the separate headings of 'Postulates' and 'Axioms.'*

They are here printed in italics, in order to keep them distinct from the Definitions.

All interpolated matter will be enclosed in square brackets.]

§ 1.

A **Point** has [position but] no magnitude.

§ 2.

A **Line** has [position and] length, but no breadth or thickness.

§ 3. Axiom.

The extremities of a finite Line are Points.

§ 4.

A **Superficies** has [position,] length, and breadth, but no thickness.

§ 5.

[A finite Superficies is called **a Figure.**]

§ 6. AXIOM.

The boundaries of a Figure are Lines.

§ 7.

A right Line is one that lies evenly as to Points in it.

§ 8. POSTULATE.

A right Line can be drawn from any Point to any other Point.

§ 9. AXIOM.

(In 3 equivalent forms.)

(1). [*Only one such line can be drawn.*]

(2). (EUCLID'S form) *Two right Lines cannot enclose a Superficies.*

(3). [*Two right Lines, which coincide at two different Points, coincide between them.*]

§ 10. POSTULATE.

A right Line, terminated at a Point, can be produced beyond it.

§ 11. AXIOM.

(In 3 equivalent forms.)

(1). [*Only one such produced portion can be drawn.*]

(2). [*Two different right Lines cannot have a common segment.*]

(3). [*Two right Lines, which coincide at two different Points, coincide if produced beyond them.*]

§ 12.

A plane Superficies, or **a Plane**, is such that, whatever two Points in it be taken, the right Line passing through them lies wholly in it.

§ 13.

An angle is the declination of two right Lines from each other, which are terminated at the same Point but are not in the same right Line. [The Lines are called its **Arms**, and the Point its **Vertex.**]

§ 14.

A Figure, bounded by right Lines, is called **rectilinear.** [The bounding Lines are called its **Sides**, and the vertices of its angles are called its **Vertices.** One of the sides is sometimes called the **Base.**]

§ 15.

A plane rectilinear Figure with three Sides is called **a trilateral Figure**, or **a Triangle**; with four, **a quadrilateral Figure.**

§ 16.

If two opposite Vertices of a quadrilateral Figure be joined; the joining Line is called a **Diagonal** of the Figure.

§ 17.

If a plane rectilinear Figure have all its Sides equal, it is called **equilateral**: if all its angles, **equiangular.**

§ 18.

A Triangle with two Sides equal, is called **isosceles**; with all unequal, **scalene.**

§ 19.

A **Circle** is a plane Figure bounded by one Line, and such that all right Lines, drawn from a certain Point within it to the bounding Line, are equal. The bounding Line is called its **Circumference,** and the Point its **Centre.**

§ 20. POSTULATE.

A Circle can be described about any Centre, and at any distance from it [*i.e. so that its Circumference shall pass through any given Point*].

§ 21.

When a right Line, meeting another, makes the adjacent angles equal, each of them is called **a right angle,** [and the first Line is said to be **at right angles to,** or **perpendicular to,** the other].

§ 22. AXIOM.

All right angles are equal. [*See Appendix* A. § 2.

§ 23.

If an angle be greater than a right angle, it is said to be **obtuse**; if less, **acute.**

§ 24.

Lines which, being in the same Plane, do not meet, however far produced, are said to be **parallel** to each other.

§ 25.

A **Parallelogram** is a quadrilateral Figure whose opposite sides are parallel.

§ 26.

[A Parallelogram is said to be **about** any Line which passes through two opposite vertices.]

§ 27.

[If, through a Point in a Diagonal of a Parallelogram, Lines be drawn parallel to the Sides: of the four Parallelograms so formed, the two which are not about the Diagonal are called the **Complements.**]

§ 28.

A Parallelogram, having all its angles right, is called a **rectangular Parallelogram,** or a **Rectangle.**

§ 29.

A Square is an equilateral Rectangle.

§ 30.

[If a certain Line be given, the phrase "**the square of the Line**" denotes the magnitude of any Square which has each of its sides equal to the Line.]

§ 31.

If a Triangle have one angle right, it is called **right-angled**; if one obtuse, **obtuse-angled**; if all acute, **acute-angled.**

§ 32.

In a right-angled Triangle, the side opposite to the right angle is called the **Hypotenuse.**

POSTULATES.

§ 1.

A right Line can be drawn from any Point to any other Point.

§ 2.

A right Line, terminated at a Point, can be produced beyond it.

§ 3.

A Circle can be described about any Centre, and at any distance from it [i.e. so that its Circumference shall pass through any given Point].

AXIOMS.

I. Axioms of Magnitude.

§ 1.

Things which are equal to the same are equal to one another.

§ 2.

If equals be added to equals, the wholes are equal.

§ 3.

If equals be taken from equals, the remainders are equal.

§ 4.

If equals be added to unequals, the wholes are unequal.

§ 5.

If equals be taken from unequals, the remainders are unequal.

§ 6.

Things which are doubles of the same are equal.

§ 7.

Things which are halves of the same are equal.

§ 8.

A whole is greater than a part.

§ 9.

[A thing, which is greater than one of two equals, is greater than a thing which is less than the other.]

II. Geometrical Axioms.

§ 10.

The extremities of a finite Line are Points.

§ 11.

The boundaries of a Figure are Lines.

§ 12.

Lines, and Figures, which can be so placed as to coincide, [and angles which can be so placed that the arms of the one lie along those of the other,] are equal.

§ 13.

(*In* 3 *equivalent forms.*)

(1). [From one Point to another only one right Line can be drawn.]

(2). (Euclid's *form*) Two right Lines cannot enclose a Superficies.

(3). [Two right Lines, which coincide at two different Points, coincide between them.]

§ 14.

(*In* 3 *equivalent forms.*)

(1). [If a right Line, terminated at a Point, be produced beyond it, only one such produced portion can be drawn.]

(2). [Two different right Lines cannot have a common segment.]

(3). [Two right Lines, which coincide at two different Points, coincide if produced beyond them.] [*Appendix* A. § 1.

§ 15.

All right angles are equal. [*Appendix* A. § 2.

§ 16.

If a right Line, meeting two others, make two interior angles on one side of it together less than two right angles: these two Lines, produced if necessary, will meet on that side.

CONVENTIONS.

§ 1.

All Points, Lines, and Figures, hereafter discussed, will be considered to be in one and the same Plane.

§ 2.

The word 'Line' will mean 'right Line.'

§ 3.

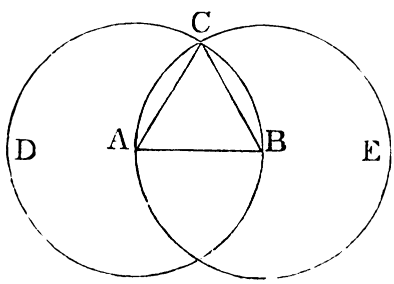
The word 'Circle' will mean 'Circumference of Circle,' whenever it is obvious that the Circumference is intended, e. g. when Circles are said to 'intersect.'

§ 4.

The words 'because,' 'therefore,' will be represented by the symbols $\because$, $\therefore$.

PROP. I. PROBLEM.

On a given Line to describe an equilateral Triangle.



Let AB be given Line. It is required to describe on it an equilateral Triangle.

About centre A, at distance AB, describe Circle BCD; about centre B, at distance BA, describe Circle ACE; and join C, where they intersect, to A and B. It is to be proved that the Triangle ABC is equilateral.

$\because$ A is centre of Circle BCD,

$\therefore$ AC is equal to AB; [Def. § 19.

also, $\because$ B is centre of Circle ACE,

$\therefore$ BC is equal to AB.

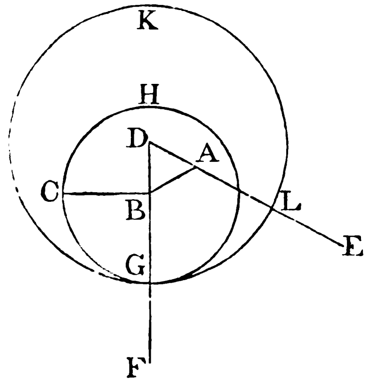
$\therefore$ Triangle ABC is equilateral;

and it is described on given Line AB.

Q.E.F.

PROP. II. PROBLEM.

From a given Point to draw a Line equal to a given Line.



Let A be given Point, and BC given Line. It is required to draw from A a Line equal to BC.

Join AB; on AB describe equilateral Triangle ABC; produce DA, DB, to E and F; about centre B, at distance BC, describe Circle CGH, cutting DF at G; and about centre D, at distance DG, describe Circle GLK, cutting DE at L. It is to be proved that AL is equal to BC.

$\because$ B is centre of Circle CGH,

$\therefore$ BC is equal to BG; [Def. § 19.

also, $\because$ D is centre of Circle GLK,

$\therefore$ DG is equal to DL;

but DB, DA, parts of these, are equal;

$\therefore$ the remainders, BG, AL, are equal; [Ax. § 3.

but BG is equal to BC;

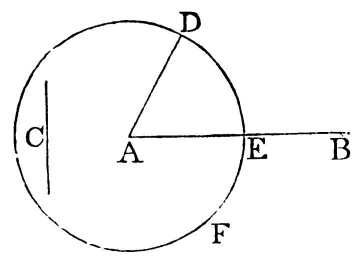
$\therefore$ AL is equal to BC, the given Line; [Ax. § 1.

and it is drawn from A, the given Point.

Q.E.F.

PROP. III. PROBLEM.

From the greater of two given unequal Lines to cut off a part equal to the less.



Let AB, C, be given Lines, of which AB is the greater. It is required to cut off from AB a part equal to C.

From A draw AD equal to C; and about centre A, at distance AD, describe Circle DEF, cutting AB at E. It is to be proved that AE is equal to C. [Prop. 2.

∵ A is centre of Circle DEF,

∴ AE is equal to AD; [Def. § 19.

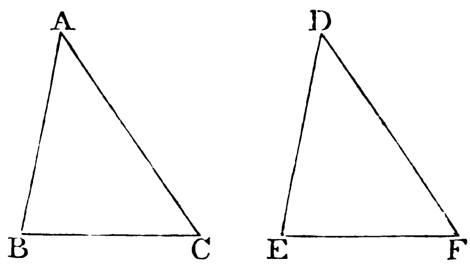
∴ it is equal to C, the given lesser Line; [Ax. § 1.

and it is cut off from AB, the given greater Line.

Q.E.F.

PROP. IV. THEOREM.

If, in two Triangles, two sides and the included angle of the one be respectively equal to two sides and the included angle of the other: then the base and the remaining angles of the one are respectively equal to the base and the remaining angles of the other, those angles being equal which are opposite to equal sides; and the Triangles are equal.



Let ABC, DEF be two Triangles, in which AB, AC are respectively equal to DE, DF, and angle A is equal to angle D. It is to be proved that BC is equal to EF; that angles B, C, are respectively equal to angles E, F; and that the Triangles are equal.

If Triangle ABC be applied to Triangle DEF, so that A may fall on D, and AB along DE,

then, $\because$ AB is equal to DE,

$\therefore$ B falls on E;

and $\because$ angle A is equal to angle D,

$\therefore$ AC falls along DF;

and $\because$ AC is equal to DF,

$\therefore$ C falls on F;

and $\because$ B falls on E, and C on F,

$\therefore$ BC coincides with EF;

$\therefore$ BC is equal to EF; [Ax. § 12.

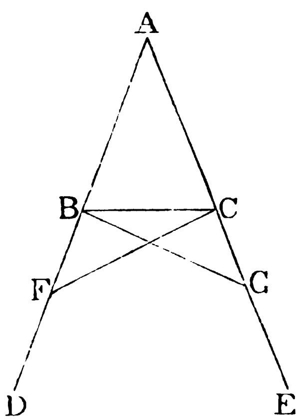
also remaining angles coincide, and therefore are equal; and Triangles coincide, and therefore are equal.

Therefore if, in two Triangles, two sides, &c. Q.E.D.

[*Appendix* A. § 1.

PROP. V. THEOREM.

The angles at the base of an isosceles Triangle are equal; and, if the equal sides be produced, the angles at the other side of the base are equal.



Let ABC be an isosceles Triangle, having AB, AC, equal; and let them be produced to D and E. It is to be proved that angles ABC, ACB are equal, and also angles DBC, ECB.

In BD take any Point F; from AE cut off AG equal to AF; and join BG, CF.

In Triangles ABG, ACF,

$\because$ $\begin{cases} AB, AG, \text{ are respectively equal to } AC, AF, \\ \text{and angle } A \text{ is common,} \end{cases}$

$\therefore$ $\begin{cases} BG \text{ is equal to } CF \\ \text{and angles } ABG, AGB, \text{ are respectively equal to} \\ \qquad \text{angles } ACF, AFC. \end{cases}$

[Prop. 4.

Next, $\because \begin{cases} AF \text{ is equal to } AG, \\ \text{and } AB,\ AC, \text{ parts of them, are equal,} \end{cases}$

$\therefore$ remainders BF, CG, are equal. [Ax. § 3.

Next, in Triangles BFC, CGB,

$\because \begin{cases} BF,\ FC, \text{ are respectively equal to } CG,\ GB, \\ \text{and angle } BFC \text{ is equal to angle } CGB, \end{cases}$

$\therefore$ angles FBC, FCB, are respectively equal to angles GCB, GBC. [Prop. 4.

Next, $\because \begin{cases} \text{angle } ABG \text{ is equal to angle } ACF, \\ \text{and angles } CBG,\ BCF, \text{ parts of them, are equal,} \end{cases}$

$\therefore$ remaining angles ABC, ACB, are equal; [Ax. § 3.

and these are angles at base.

Also it has been proved that the angles at the other side of base, namely DBC, ECB, are equal.

Therefore the angles &c.

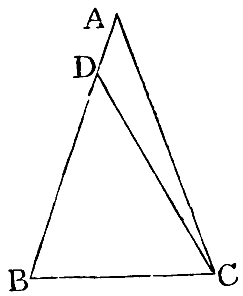
Q.E.D.

Corollary.

An equilateral Triangle is equiangular.

PROP. VI. THEOREM.

If a Triangle have two angles equal: the opposite sides are equal.



Let ABC be a Triangle, having angles B, ACB, equal. It is to be proved that AC, AB, are equal.

For if they be unequal, one must be the greater; let AB be the greater; from it cut off DB equal to AC; and join DC.

Then in Triangles DBC, ABC,

$$\because \begin{cases} DB \text{ is equal to } AC, \\ BC \text{ is common,} \\ \text{and angle } B \text{ is equal to angle } ACB, \end{cases}$$

$\therefore$ the Triangles are equal, [Prop. 4.

the part equal to the whole, which is absurd; [Ax. § 8.

$\therefore$ AC, AB, are not unequal;

i.e. they are equal.

Therefore, if a Triangle, &c.

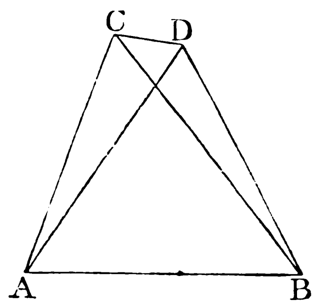
Q.E.D.

Corollary.

An equiangular Triangle is equilateral.

PROP. VII. THEOREM.

On the same base, and on the same side of it, there cannot be two different Triangles, in which the sides terminated at one end of the base are equal, and likewise those terminated at the other end.



If possible, let there be two such triangles, ABC, ABD, in which AC is equal to AD, and BC to BD.

First, let the vertex of each be without the other.

Join CD.

$\because$ AC is equal to AD,

$\therefore$ angle ADC is equal to angle ACD; [Prop. 5.

$\because$ $\begin{cases} \text{angle } BDC \text{ is greater than one of these equals,} \\ \text{and angle } BCD \text{ is less than the other,} \end{cases}$

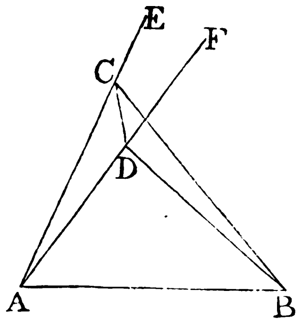
$\therefore$ angle BDC is greater than angle BCD; [Ax. § 9.

again, $\because$ BC is equal to BD,

$\therefore$ angle BDC is equal to angle BCD;

but it is also greater; which is absurd.

Secondly, let D, the vertex of one, be within the other. Join CD; and produce AC, AD, to E, F.



$\because$ AC is equal to AD,

$\therefore$ angle ECD is equal to angle FDC; [Prop 5.

$\because$ $\begin{cases} \text{angle } BDC \text{ is greater than one of these equals,} \\ \text{and angle } BCD \text{ is less than the other,} \end{cases}$

$\therefore$ angle BDC is greater than angle BCD; [Ax. § 9.

again, $\because$ BC is equal to BD,

$\therefore$ angle BDC is equal to angle BCD;

but it is also greater; which is absurd.

The case, where the vertex of one is in one side of the other, needs no demonstration.

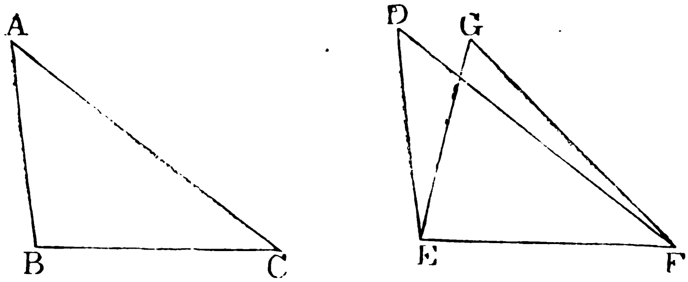
Therefore, on the same base, &c.

Q E.D.

[*Appendix* A. § 3.

PROP. VIII. Theorem.

If, in two Triangles, the sides of the one be respectively equal to the sides of the other: then the angles of the one are respectively equal to the angles of the other, those angles being equal which are opposite to equal sides.



Let ABC, DEF be two Triangles, in which AB, AC, BC, are respectively equal to DE, DF, EF. It is to be proved that angles A, B, C, are respectively equal to angles D, E, F.

If Triangle ABC be applied to Triangle DEF, so that B may fall on E, and BC along EF,

then, ∵ BC is equal to EF,

∴ C falls on F;

now, if BA, AC did not fall on ED, DF, but had another position, as for instance EG, GF, there would be on the same base, and on the same side of it, two different Triangles, in which the sides terminated at one end of the base would be equal, and likewise those terminated at the other end; [Prop. 7.

but this is impossible;

∴ BA, AC, fall on ED, DF;

∴ angle A coincides with angle D;

∴ it is equal to angle D; [Ax. § 12.

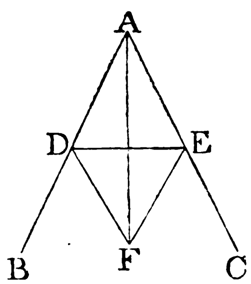
and similarly for the other angles.

Therefore if, in two Triangles, &c. Q.E.D.

[*Appendix* A. § 4.

PROP. IX. Problem.

To bisect a given angle.



Let BAC be given angle. It is required to bisect it.

In AB take any Point D; from AC cut off AE equal to AD; join DE; on DE, on side remote from A, describe equilateral Triangle DFE; and join AF. It is to be proved that angle BAC is bisected by AF.

In Triangles ADF, AEF,

$\because \begin{cases} AD \text{ is equal to } AE, \\ AF \text{ is common,} \\ \text{and } DF \text{ is equal to } EF, \end{cases}$

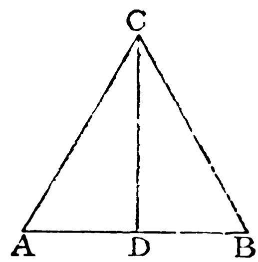
$\therefore$ angle DAF is equal to angle EAF; [Prop. 8.

i. e. angle BAC is bisected by AF.

Q.E.F.

PROP. X. PROBLEM.

To bisect a given Line.



Let AB be given Line. It is required to bisect it.

On AB describe equilateral Triangle ACB; and bisect angle ACB by Line CD, meeting AB at D. It is to be proved that AB is bisected at D.

In Triangles CAD, CBD,

$\because \begin{cases} CA \text{ is equal to } CB, \\ CD \text{ is common,} \\ \text{and angle } ACD \text{ is equal to angle } BCD, \end{cases}$

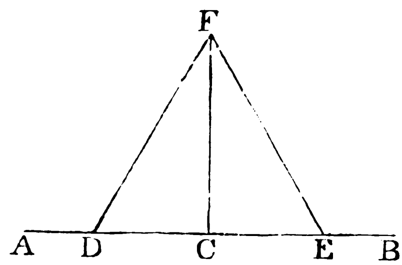
$\therefore AD$ is equal to DB; [Prop. 4.

i. e. Line AB is bisected at D.

Q.E.F.

PROP. XI. PROBLEM.

From a given Point in a given Line to draw a Line at right angles to it.



Let AB be given Line, and C given Point in it. It is required to draw a Line from C, at right angles to AB.

In AC take any point D; from CB cut off CE equal to CD; on DE describe equilateral Triangle DFE; and join CF. It is to be proved that CF is at right angles to AB.

In Triangles FDC, FEC,

$\because \begin{cases} CD \text{ is equal to } CE, \\ CF \text{ is common,} \\ \text{and } FD \text{ is equal to } FE, \end{cases}$ [Def. § 17.

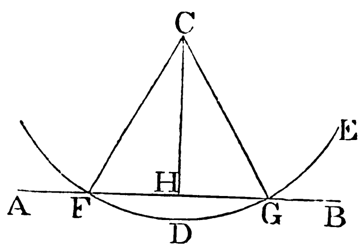
$\therefore$ angle FCD is equal to angle FCE; [Prop. 8.

i.e. CF is drawn at right angles to AB. [Def. § 21.

Q.E.F.

PROP. XII. Problem.

From a given Point without a given Line to draw a Line at right angles to it.



Let AB be given Line, and C given Point without it. It is required to draw a Line from C, at right angles to AB.

On the other side of AB take any point D; about centre C, at distance CD, describe Circle EGF, cutting AB at F, G; join CF, CG; and bisect angle FCG by Line CH. It is to be proved that CH is at right angles to AB.

In Triangles CFH, CGH,

$\because \begin{cases} CF \text{ is equal to } CG, \\ CH \text{ is common,} \\ \text{and angle } FCH \text{ is equal to angle } GCH, \end{cases}$ [Def. § 19.

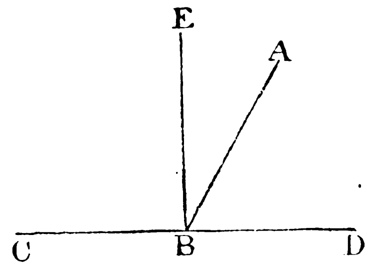
$\therefore$ angle CHF is equal to angle CHG; [Prop. 4.

i. e. CH is drawn at right angles to AB. [Def. § 21.

Q.E.F.

PROP. XIII. THEOREM.

Any two adjacent angles, which one Line makes with another, are together equal to two right angles.



Let AB make, with CD, angles CBA, ABD. It is to be proved that they are together equal to two right angles.

If angle CBA be equal to angle ABD, each is right.

If not, draw BE at right angles to CD.

Then angles CBA, ABD, are together equal to angles CBE, EBA, ABD,

i. e. to angles CBE, EBD,

i. e. to two right angles.

Therefore any two adjacent angles, &c.

Q.E.D.

COROLLARIES.

1. *The angles made on one side of a Line by any number of Lines meeting at a Point in it are together equal to two right angles.*

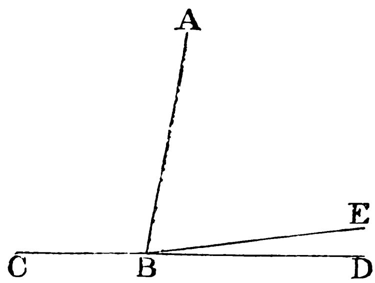
2. *The angles made all round a Point by any number of Lines meeting at it are together equal to four right angles.*

3. [*If one of two adjacent angles, which one Line makes with another, be right: so also is the other.*]

4. [*If one of the four angles, made by two intersecting Lines, be right: so also are the others.*]

PROP. XIV. Theorem.

If, at a Point in a Line, two other Lines, on opposite sides of it, make adjacent angles together equal to two right angles: they are in the same Line.



At Point *B* in Line *AB*, let Lines *CB*, *BD*, on opposite sides of it, make adjacent angles *CBA*, *ABD*, together equal to two right angles. It is to be proved that *CB*, *BD*, are in the same Line.

If not, from *B* draw *BE* in same Line with *CB*.

Then angles *CBA*, *ABE*, are together equal to two right angles; [Prop. 13.

but angles *CBA*, *ABD*, are together equal to two right angles; [Hyp.

∴ angles *CBA*, *ABE*, are together equal to angles *CBA*, *ABD*. [Ax. §§ 1, 15.

From each of these equals take common angle *CBA*;

∴ remaining angles *ABE*, *ABD* are equal, the part equal to the whole, which is absurd.

∴ *BE* is not in same Line with *CB*; and the same may be proved for any other Line drawn from *B*, except *BD*;

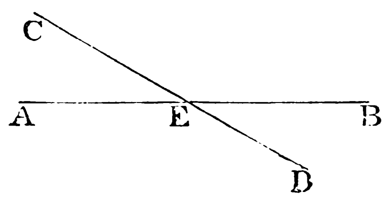
∴ *BD* is in same Line with *CB*.

Therefore if, at a Point &c.

Q.E.D.

PROP. XV. THEOREM.

If two Lines intersect: the vertical angles are equal.



Let AB, CD be two Lines intersecting at E. It is to be proved that angles AEC, BED are equal; and also angles AED, BEC.

Angles AEC, CEB, are together equal to two right angles; [Prop. 13.

and angles CEB, BED, are together equal to two right angles;

∴ angles AEC, CEB, are together equal to angles CEB, BED; [Ax. §§ 1, 15.

from each of these equals take common angle CEB;

∴ remaining angles AEC, BED are equal.

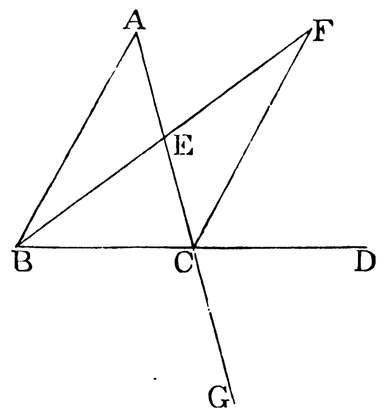
Similarly it may be proved that angles AED, BEC, are equal.

Therefore, if two Lines &c.

Q.E.D.

PROP. XVI. THEOREM.

If one side of a Triangle be produced, the exterior angle is greater than either of the interior opposite angles.



Let ABC be a Triangle, having side BC produced to D. It is to be proved that angle ACD is greater than either of angles A, ABC.

Bisect AC at E; join BE; produce BE to F, making EF equal to BE; and join FC.

In Triangles ABE, FEC,

$\because \begin{cases} AE, EB, \text{ are respectively equal to } CE, EF, \\ \text{and angle } AEB \text{ is equal to angle } FEC, \end{cases}$ [Prop. 15.

$\therefore$ angle A is equal to angle ECF; [Prop. 4.

but angle ACD is greater than angle ECF; [Ax. § 8.

$\therefore$ it is greater than angle A.

Hence, if AC be produced to G, angle BCG is greater than angle ABC;

but angle ACD is equal to angle BCG; [Prop. 15.

$\therefore$ it is greater than angle ABC.

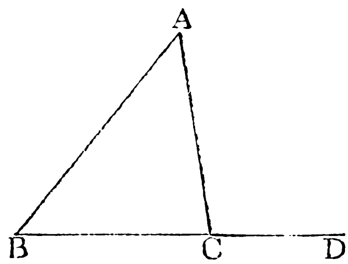
Therefore, if one side of a Triangle &c.

Q.E.D.

[*Appendix* A. § 5.

PROP. XVII. THEOREM.

Any two angles of a Triangle are together less than two right angles.



Let ABC be a Triangle.

Produce BC to D.

Then exterior angle ACD is greater than interior opposite angle B; [Prop. 16.

to each of these unequals add angle ACB;

$\therefore$ angles ACD, ACB, are together greater than angles B, ACB;

but angles ACD, ACB, are together equal to two right angles. [Prop. 13.

$\therefore$ angles B, ACB, are together less than two right angles.

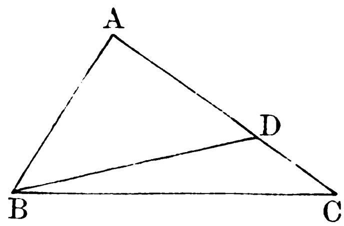
Similarly it may be proved that any other two of the angles are together less than two right angles.

Therefore any two angles &c.

Q.E.D.

PROP. XVIII. THEOREM.

The greater side of a Triangle is opposite to the greater angle.



Let ABC be a Triangle having AC greater than AB.
It is to be proved that angle ABC is greater than angle C.

From AC cut off AD equal to AB; and join BD.

$\because$ AB is equal to AD,

$\therefore$ angle ABD is equal to angle ADB; [Prop. 5.

$\because$ $\left\{\begin{array}{l}\text{angle } ABC \text{ is greater than one of these equals,}\\ \text{and angle } C \text{ is less than the other,}\end{array}\right.$ [Prop. 16.

$\therefore$ angle ABC is greater than angle C. [Ax. § 20.

Therefore the greater side &c.

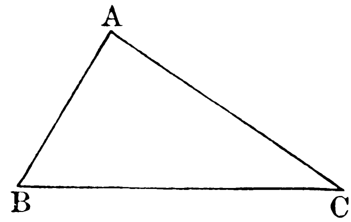
Q.E.D.

[*Note.* The enunciation may also be expressed thus:—

If a Triangle have one side greater than another: the angle opposite to the one is greater than the angle opposite to the other.]

PROP. XIX. THEOREM.

The greater angle of a Triangle is opposite to the greater side.



Let ABC be a Triangle having angle B greater than angle C. It is to be proved that AC is greater than AB.

For if not, it must be equal or less.

But it is not equal, for then angle B would be equal to angle C; [Prop. 5.

neither is it less, for then angle B would be less than angle C; [Prop. 18.

$\therefore$ it is greater.

Therefore the greater angle &c.

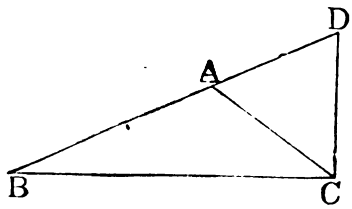
Q.E.D.

[*Note.* The enunciation may also be expressed thus —

If a Triangle have one angle greater than another: the side subtending, i.e. opposite to, the one is greater than the side subtending the other.]

PROP. XX. THEOREM.

Any two sides of a Triangle are together greater than the third.



Let ABC be a Triangle.

Produce BA to D, making AD equal to AC; and join DC.

Then, $\because$ AD is equal to AC,

$\therefore$ angle ACD is equal to angle D; [Prop. 5.

but angle BCD is greater than angle ACD;

$\therefore$ it is also greater than angle D;

$\therefore$ BD is greater than BC; [Prop. 19.

but BA, AC, are together equal to BD;

$\therefore$ BA, AC, are together greater than BC.

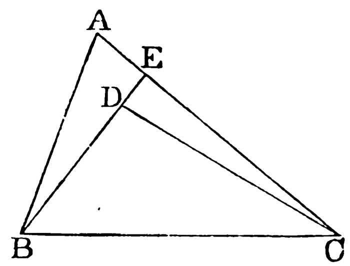
Similarly it may be proved that any other two of the sides are together greater than the third.

Therefore any two sides &c.

Q.E.D.

PROP. XXI. THEOREM.

If, from the ends of one side of a Triangle, Lines be drawn to a Point within it: they are together less than the other two sides, but contain a greater angle.



Let ABC be a Triangle; and from B, C, let BD, CD be drawn to a Point D within it. It is to be proved that BD, DC, are together less than BA, AC; but that angle BDC is greater than angle A.

Produce BD to meet AC in E.

Then, $\because$ BA, AE, are together greater than BE; [Prop. 20.

to each of these unequals add EC;

$\therefore$ BA, AC, are together greater than BE, EC.

Similarly, $\because$ DE, EC, are together greater than DC;

to each of these unequals add BD;

$\therefore$ BE, EC, are together greater than BD, DC;

$\therefore$, *a fortiori*, BA, AC, are together greater than BD, DC.

Again, $\because$ CED is exterior angle of Triangle ABE,

$\therefore$ it is greater than angle A; [Prop. 16.

similarly angle BDC is greater than angle CED;

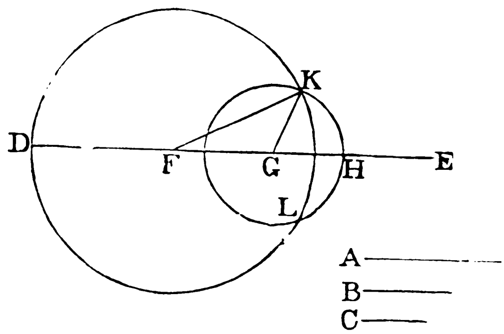
$\therefore$, *a fortiori*, angle BDC is greater than angle A.

Therefore if, from the ends &c.

Q.E.D.

PROP. XXII. PROBLEM.

To describe a Triangle having its sides respectively equal to three given Lines, of which any two are greater than the third.



Let A, B, C, be given Lines: it is required to describe a Triangle having its sides respectively equal to A, B, C.

Take a Line DE, terminated at D, but unlimited towards E; from it cut off DF equal to A, FG equal to B, GH equal to C; about centre F, at distance FD, describe Circle DKL; about centre G, at distance GH, describe Circle KHL; and join K, where the Circles intersect, to F and G.

Then, $\because$ F is centre of Circle DKL,

$\therefore$ FK is equal to FD; [Def. § 19.

i.e. to A.

Similarly GK is equal to GH, i.e. to C.

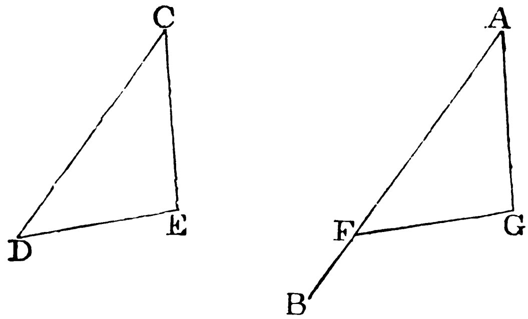
And FG was made equal to B.

Therefore Triangle FGK has its sides respectively equal to given Lines A, B, C.

Q.E.F.

PROP. XXIII. PROBLEM.

At a given Point in a given Line to make an angle equal to a given angle.



Let AB be given Line, A given Point, and DCE given angle. It is required to make at A, in Line AB, an angle equal to angle DCE.

In CD, CE take any Points D, E, and join DE; and describe Triangle AFG having its sides respectively equal to CD, CE, DE, of which any two are (by Prop. 20) greater than the third.

Then, in Triangles AFG, CDE,

$\because$ AF, AG, FG, are respectively equal to CD, CE, DE,

$\therefore$ angle A is equal to angle C; [Prop. 8.

that is, angle A has been made at given Point A, in given Line AB, equal to given angle C.

Q.E.F.

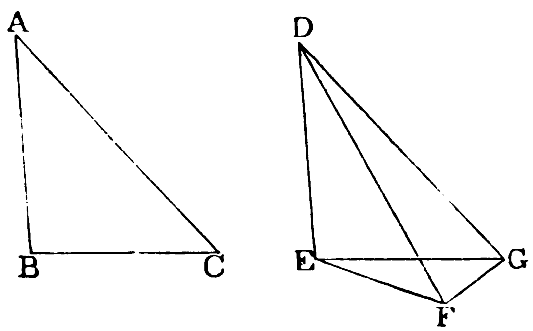
PROP. XXIV. THEOREM.

If, in two Triangles, two sides of the one be respectively equal to two sides of the other, but the included angle of the one greater than the included angle of the other: the base of the one is greater than the base of the other.

Let ABC, DEF be two Triangles, in which AB, AC, are respectively equal to DE, DF, but angle A is greater than angle EDF. It is to be proved that BC is greater than EF.

At D, in Line DE, make angle EDG equal to angle A; make DG equal to AC or DF; and join GE, GF.

First, let F be without Triangle DEG.



Then, in Triangles ABC, DEG,

$\because$ { AB, AC, are respectively equal to DE, DG, and angle A is equal to angle EDG,

$\therefore$ BC is equal to EG. [Prop. 4.

Also, $\because$ DF is equal to DG.

$\therefore$ angle DFG is equal to angle DGF; [Prop. 5.

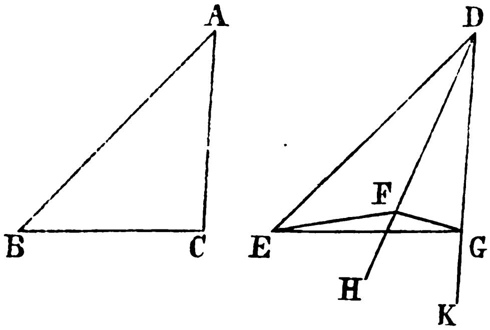
$\because$ { angle EFG is greater than one of these equals, and angle EGF is less than the other,

$\therefore$ angle EFG is greater than angle EGF; [Ax. § 9.

$\therefore$ EG is greater than EF; [Prop. 19.

$\therefore$ BC is greater than EF.

Secondly, let F be within Triangle DEG.



Produce DF, DG, to H, K.

Then it may be proved, as in the first case, that BC is equal to EG.

Also, $\because$ DF is equal to DG,

$\therefore$ angle HFG is equal to angle KGF; [Prop. 5.

$\because$ $\left\{\begin{array}{l}\text{angle } EFG \text{ is greater than one of these equals,}\\ \text{and angle } EGF \text{ is less than the other,}\end{array}\right.$

$\therefore$ angle EFG is greater than angle EGF; [Ax. § 9.

$\therefore$ EG is greater than EF; [Prop. 19.

$\therefore$ BC is greater than EF.

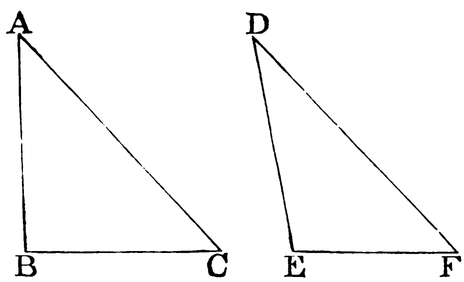
The case, in which F is in the Line EG, needs no demonstration.

Therefore if, in two Triangles, &c.

Q.E.D.

PROP. XXV. THEOREM.

If, in two Triangles, two sides of the one be respectively equal to two sides of the other, but the base of the one greater than the base of the other: the included angle of the one is greater than the included angle of the other.



Let ABC, DEF, be two Triangles, in which AB, AC, are respectively equal to DE, DF, but BC is greater than EF. It is to be proved that angle A is greater than angle D.

For if not, it must be equal or less.

But it is not equal, for then BC would be equal to EF; [Prop. 4.

neither is it less, for then BC would be less than EF; [Prop. 24.

∴ it is greater.

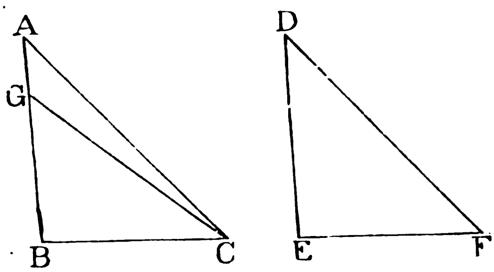
Therefore if, in two Triangles, &c.

Q.E.D.

PROP. XXVI. THEOREM.

If, in two Triangles, two angles and a side of the one be respectively equal to two angles and a side of the other; namely, either the sides adjacent to the equal angles, or sides opposite to equal angles: then the remaining sides and angle of the one are respectively equal to the remaining sides and angle of the other, those sides being equal which are opposite to equal angles.

First, let those sides be equal which are adjacent to the equal angles.



Let ABC, DEF be two Triangles, in which angles B, ACB, are respectively equal to angles E, F, and BC is equal to EF. It is to be proved that AB, AC, are respectively equal to DE, DF, and that angle A is equal to angle D.

For, if AB be not equal to DE, one must be the greater; let AB be the greater; from it cut off GB equal to DE; and join GC.

Then, in Triangles GBC, DEF,

$\because \begin{cases} GB, BC, \text{ are respectively equal to } DE, EF, \\ \text{and angle } B \text{ is equal to angle } E, \end{cases}$

$\therefore$ angle GCB is equal to angle F; [Prop. 4.

i.e. to angle ACB, the part equal to the whole, which is absurd; [Ax. § 8.

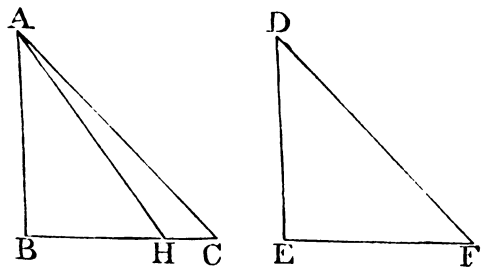
$\therefore$ AB is equal to DE.

Then, in Triangles ABC, DEF,

$\because \begin{cases} AB, BC, \text{ are respectively equal to } DE, EF, \\ \text{and angle } B \text{ is equal to angle } E, \end{cases}$

$\therefore AC$ is equal to DF, and angle A to angle D. [Prop. 4.

Secondly, let sides be equal which are opposite to equal angles.



Let ABC, DEF, be two Triangles, in which angles B, C, are respectively equal to angles E, F, and AB is equal to DE. It is to be proved that BC, CA, are respectively equal to EF, FD, and that angle BAC is equal to angle D.

For, if BC be not equal to EF, one must be the greater; let BC be the greater; from it cut off BH equal to EF; and join AH.

Then, in Triangles ABH, DEF,

$\because \begin{cases} AB, BH, \text{ are respectively equal to } DE, EF, \\ \text{and angle } B \text{ is equal to angle } E, \end{cases}$

$\therefore$ angle AHB is equal to angle F; [Prop. 4.

i.e. to angle C, the exterior equal to the interior opposite, which is absurd; [Prop. 16.

$\therefore BC$ is equal to EF.

Then it may be proved, as in the first case, that AC is equal to DF, and angle BAC to angle D. [Prop. 4.

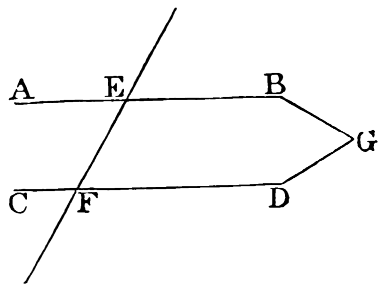
Therefore if, in two Triangles, &c.

Q.E.D.

[*Appendix* A. § 6.

PROP. XXVII. THEOREM.

If a Line, meeting two others, make two alternate angles equal: these two Lines are parallel.



Let EF, meeting AB, CD, make the alternate angles AEF, EFD equal. It is to be proved that AB, CD, are parallel.

For if not, they must meet when produced towards A, C, or else towards B, D; let them be produced and meet towards B, D, at G.

Then EGF is a Triangle, and exterior angle AEF is greater than interior opposite angle EFD; [Prop. 16.

but it is also equal to it, which is absurd;

$\therefore$ AB, CD, do not meet towards B, D.

Similarly it may be proved that they do not meet towards A, C.

Therefore they are parallel. [Def. § 24.

Therefore, if a Line, &c.

Q.E.D.

---

from each of these equals take common angle BGH;

$\therefore$ remaining angles AGH, GHD, are equal;

and these are alternate angles;

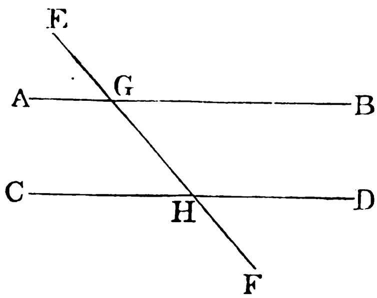
$\therefore$ AB, CD, are parallel. [Prop. 27.

Therefore, if a Line, &c. Q.E.D.

[*Appendix* A. § 7.

PROP. XXVIII. Theorem.

If a Line, cutting two others, make an exterior angle equal to the interior opposite angle on the same side of the Line; or make two interior angles on the same side of the Line together equal to two right angles: these two Lines are parallel.



Let EF cut AB, CD, at G, H.

First, let exterior angle EGB be equal to interior opposite angle GHD. It is to be proved that AB, CD, are parallel.

For angle AGH is equal to vertical angle EGB; [Prop. 15.

∴ it is also equal to angle GHD;

and these are alternate angles;

∴ AB, CD, are parallel. [Prop. 27.

Secondly, let two interior angles BGH, GHD, be together equal to two right angles: it is to be proved that AB, CD, are parallel.

For angles AGH, BGH, are together equal to two right angles; [Prop. 13.

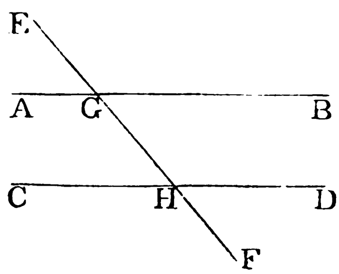
and angles BGH, GHD, are together equal to two right angles;

∴ angles AGH, BGH, are together equal to angles BGH, GHD; [Ax. §§ 1, 15.

[*continued opposite.*]

PROP. XXIX. Theorem.

If a Line cut two Parallels: it makes each pair of alternate angles equal; each exterior equal to the interior opposite on the same side of it; and each pair of interior on the same side of it together equal to two right angles.



Let EF cut the two Parallels AB, CD, in G, H. It is to be proved that alternate angles AGH, GHD, are equal; that exterior EGB is equal to interior opposite GHD; and that two interior BGH, GHD, are together equal to two right angles.

For if angle AGH be not equal to angle GHD, one must be the greater; let angle AGH be the greater, and to each of these unequals add angle BGH;

∴ angles AGH, BGH, are together greater than angles BGH, GHD;

but angles AGH, BGH, are together equal to two right angles; [Prop. 13.

∴ angles BGH, GHD, are together less than two right angles;

∴ AB, CD, produced if necessary, will meet; [Ax. § 16.
that is, Parallels will meet, which is absurd;

∴ angle AGH is not unequal to angle GHD;

i.e. angle AGH is equal to angle GHD;

and these are alternate angles.

Again, angle EGB is equal to vertical angle AGH; [Prop. 15.

∴ it is also equal to angle GHD;

and these angles are exterior and interior opposite on the same side.

To each of these equals add angle BGH;

∴ angles BGH, GHD, are together equal to angles EGB, BGH;

i. e. to two right angles;

and these are two interior angles on the same side.

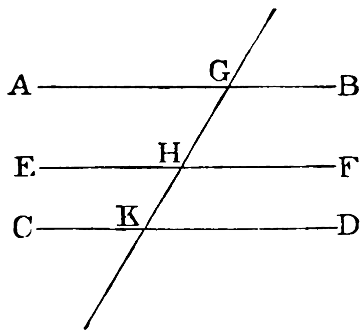
Similar conclusions may be proved for the other angles.

Therefore, if a Line &c.

Q.E.D.

PROP. XXX. THEOREM.

Two different Lines, which are parallel to the same Line, are parallel to each other.



First, let the two Lines be on the same side of the third.

Let AB, EF, be both parallel to CD. It is to be proved that AB is parallel to EF.

Draw a Line meeting the three Lines at G, H, K.

Then, $\because$ GK meets Parallels AB, CD,

$\therefore$ angle AGK is equal to alternate angle GKD; [Prop. 29.

Also, $\because$ GK cuts Parallels EF, CD,

$\therefore$ exterior angle GHF is equal to interior opposite angle GKD; [Prop. 29.

$\therefore$ angle AGH is equal to angle GHF; [Ax. § 1

and these are alternate angles;

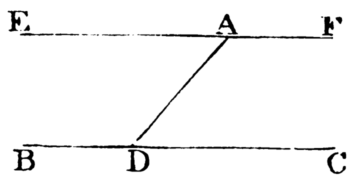
$\therefore$ AB is parallel to EF. [Prop. 27.

The case, in which the two Lines are on opposite sides of the third, needs no demonstration.

Therefore two Lines &c. Q.E.D.

PROP. XXXI. PROBLEM.

Through a given Point without a given Line to draw a Line parallel to it.



Let A be given Point, and BC given Line. It is required to draw through A a Line parallel to BC.

In BC take any Point D; join AD; at A, in Line AD, make angle DAE equal to angle ADC; and produce EA to F. It is to be proved that EF is parallel to BC.

$\because$ angle EAD is equal to alternate angle ADC,

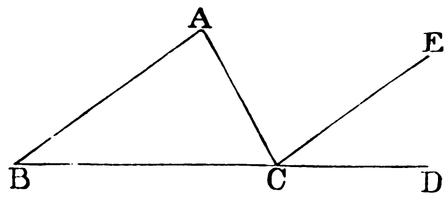
$\therefore$ EF is parallel to BC; [Prop. 27.

and it is drawn through given Point A.

Q.E.F.

PROP. XXXII. THEOREM.

If a side of a Triangle be produced: the exterior angle is equal to the two interior opposite angles; and the three interior angles are together equal to two right angles.



Let ABC be a Triangle; and let BC be produced to D. It is to be proved that exterior angle ACD is equal to two interior opposite angles A, B; and that three interior angles A, B, ACB, are together equal to two right angles.

From C draw CE parallel to AB.

∵ AC meets Parallels AB, EC,

∴ angle ACE is equal to alternate angle A; [Prop. 29.

and, ∵ BD cuts them,

∴ exterior angle ECD is equal to interior opposite angle B; [Prop. 29.

∴ angles ACE, ECD, are together equal to angles A, B;

i.e. angle ACD is equal to angles A, B.

To each of these equals add angle ACB;

∴ angles A, B, ACB, are together equal to angles ACD, ACB;

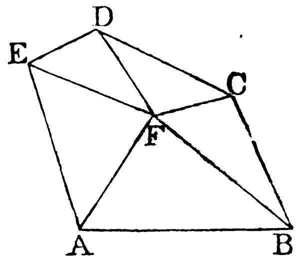
i.e. are together equal to two right angles; [Prop. 13.

Therefore if a side &c.

Q.E.D.

Corollary 1.

The interior angles of a rectilinear Figure, with four right angles, are equal to twice as many right angles as the Figure has sides.



Let $ABCDE$ be a rectilinear Figure.

Take a Point F within it, and draw Lines from F to all vertices.

Then Figure is divided into as many Triangles as it has sides.

∵ angles of each Triangle are together equal to two right angles,

∴ angles of all the Triangles are together equal to twice as many right angles as Figure has sides;

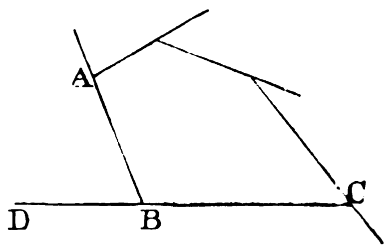
but these angles are together equal to interior angles of Figure, with angles about F, that is, with four right angles;

∴ interior angles of Figure, with four right angles, are together equal to twice as many right angles as Figure has sides.

Q.E.D.

COROLLARY 2.

If each side of a rectilinear Figure be produced; the exterior angles are together equal to four right angles.



∵ each exterior angle, with its adjacent interior angle, is equal to two right angles, [Prop. 13.

∴ all the exterior angles, with all the interior angles, are together equal to twice as many right angles as Figure has sides;

i.e. to the interior angles with four right angles; [Cor. 1.

from each of these equals take the interior angles;

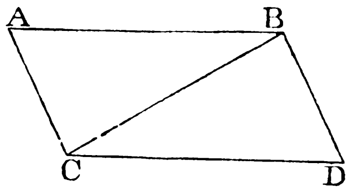
∴ all the exterior angles are together equal to four right angles.

Q.E.D.

[*Appendix* A. § 8.

PROP. XXXIII. Theorem.

The Lines which join the ends of two equal and parallel Lines, towards the same parts, are themselves equal and parallel.



Let AB, CD, be equal and parallel, and let their ends be joined by AC, BD. It is to be proved that AC, BD, are equal and parallel.

Join BC.

$\because$ BC meets Parallels AB, CD,

$\therefore$ angle ABC is equal to alternate angle BCD; [Prop. 29.

then, in Triangles ABC, BCD,

$\because \begin{cases} AB \text{ is equal to } CD, \\ BC \text{ is common,} \\ \text{and angle } ABC \text{ is equal to angle } BCD, \end{cases}$

$\therefore \begin{cases} AC \text{ is equal to } BD, \\ \text{and angle } ACB \text{ is equal to angle } CBD; \end{cases}$ [Prop. 4.

but these are alternate angles;

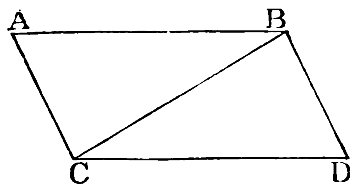
$\therefore$ AC is parallel to BD. [Prop. 27.

Therefore the Lines which join &c.

Q.E.D.

PROP. XXXIV. THEOREM.

The opposite sides and angles of a Parallelogram are equal; and each Diagonal bisects it.



Let $ABDC$ be a Parallelogram, and BC a Diagonal of it. It is to be proved that AB is equal to CD, AC to BD, angle A to angle D, angle ABD to angle ACD, and that BC bisects the Parallelogram.

$\because$ BC meets Parallels AB, CD,

$\therefore$ angle ABC is equal to alternate angle BCD; [Prop. 29.

and, $\because$ BC meets Parallels AC, BD,

$\therefore$ angle ACB is equal to alternate angle CBD; [Prop. 29.

whence also, whole angle ABD is equal to whole angle ACD.

Also, in Triangles ABC, BCD,

$\therefore \begin{cases} \text{angles } ABC, ACB, \text{ are respectively equal to angles } BCD, CBD. \\ \text{and } BC \text{ is common,} \end{cases}$

$\therefore \begin{cases} AB, AC, \text{ are respectively equal to } CD, BD, \\ \text{and angle } A \text{ is equal to angle } D; \end{cases}$ [Prop. 26.

whence also Triangles are equal; [Prop. 4.

i.e. BC bisects Parallelogram;

similarly, if A, D, be joined, it may be proved that AD bisects the Parallelogram.

Therefore the opposite sides &c.

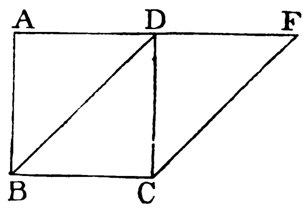
Q.E.D.

PROP. XXXV. THEOREM.

Parallelograms on the same base and between the same parallels are equal.

Let Parallelograms AC, BF, be on same base BC, and between same parallels AF, BC. It is to be proved that they are equal.

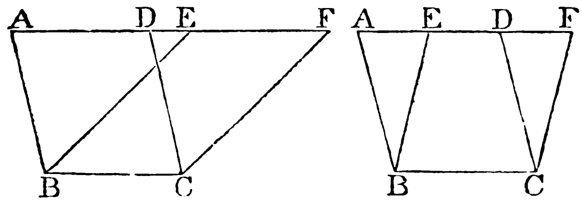
First, let sides opposite to base be terminated at same Point D.



∵ each Parallelogram is double of Triangle DBC;

∴ they are equal. [Ax. § 6.

Next, let sides opposite to base be not terminated at same Point.



∵ AC is a Parallelogram,

∴ AD is equal to BC; [Prop. 34.

similarly, EF is equal to BC;

∴ AD is equal to EF;

to each add, or from each subtract, DE;

∴ AE is equal to DF.

Also, ∵ AC is a Parallelogram,

∴ AB is equal to DC.

Also, ∵ FA meets Parallels AB, DC,

∴ exterior angle FDC is equal to interior opposite angle A.

Then, in Triangles ABE, DCF,

∵ $\left\{\begin{array}{l}AB, AE, \text{ are respectively equal to } DC, DF, \\ \text{and angle } A \text{ is equal to angle } FDC,\end{array}\right.$

∴ the Triangles are equal; [Prop. 4.

∴, if each be taken from Figure $ABCF$, the remainders are equal;

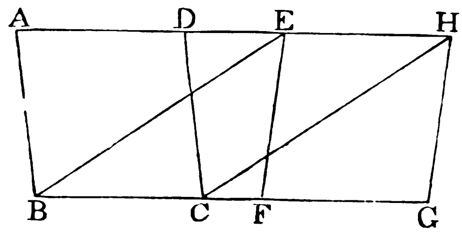
i.e. Parallelograms BF, AC, are equal.

Therefore Parallelograms on the same base &c.

Q.E.D.

PROP. XXXVI. THEOREM.

Parallelograms on equal bases and between the same Parallels are equal.



Let Parallelograms AC, EG, be on equal bases BC, FG, and between same Parallels AH, BG. It is to be proved that they are equal.

Join BE, CH.

$\because \begin{cases} BC \text{ is equal to } FG, & \text{[Hyp.} \\ \text{and } EH \text{ is equal to } FG, & \text{[Prop. 34.} \end{cases}$

$\therefore$ BC is equal to EH;

Also they are parallel;

$\therefore$ BE is parallel to CH; [Prop. 33.

$\therefore$ $EBCH$ is a Parallelogram.

Next, $\because$ Parallelograms AC, EC, are on same base and between same Parallels,

$\therefore$ they are equal; [Prop. 35.

similarly Parallelograms EC, EG are equal.

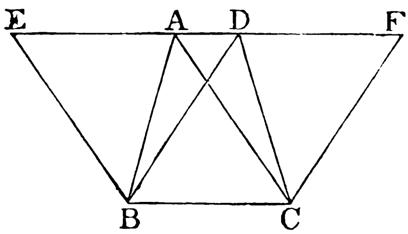
$\therefore$ Parallelograms AC, EG are equal.

Therefore Parallelograms on equal bases &c.

Q.E.D.

PROP. XXXVII. THEOREM.

Triangles on the same base and between the same parallels are equal.



Let Triangles ABC, DBC, be on same base BC, and between same Parallels AD, BC. It is to be proved that they are equal.

Produce AD to E and F; from B draw BE parallel to CA; and from C draw CF parallel to BD.

Then Figures EC, FB, are Parallelograms.

Also they are on same base and between same Parallels;

$\therefore$ they are equal. [Prop. 35.

But Triangles ABC, DBC, are halves of them; [Prop. 34.

$\therefore$ they also are equal. [Ax. § 7.

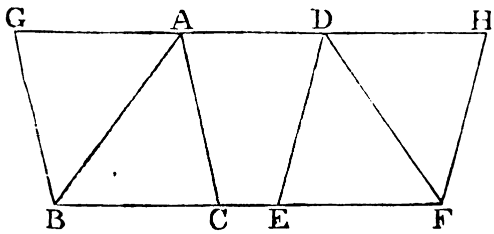
Therefore Triangles on the same base &c.

Q.E.D.

[*Appendix* A. § 9.

PROP. XXXVIII. Theorem.

Triangles on equal bases and between the same Parallels are equal.



Let Triangles *ABC*, *DEF*, be on equal bases *BC*, *EF*, and between same Parallels *AD*, *BF*. It is to be proved that they are equal.

Produce *AD* to *G* and *H*; from *B* draw *BG* parallel to *CA*; and from *F* draw *FH* parallel to *ED*.

Then the Figures *GC*, *HE*, are Parallelograms.

Also they are on equal bases and between same Parallels;

∴ they are equal. [Prop. 36.

But Triangles *BAC*, *DEF*, are halves of them; [Prop. 34.

∴ they also are equal. [Ax. § 7.

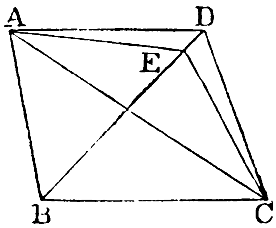
Therefore Triangles on equal bases &c.

Q.E.D.

[*Appendix* A. § 10.

PROP. XXXIX. Theorem.

Equal Triangles on the same base, and on the same side of it, are between the same Parallels.



Let equal Triangles ABC, DBC, be on same base BC, and on same side of it. It is to be proved that they are between same Parallels.

Join AD.

Now, if AD be not parallel to BC, from A draw AE parallel to BC, and meeting BD, or BD produced, at E; and join EC; and first, let AE fall below AD.

$\because$ Triangles ABC, EBC, are on same base and between same Parallels,

$\therefore$ Triangle EBC is equal to Triangle ABC; [Prop. 37.

i. e. to Triangle DBC, the part equal to the whole, which is absurd.

Similarly, if AE fall above AD.

$\therefore$ AD is parallel to BC.

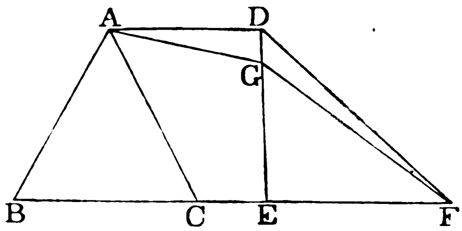
Therefore equal Triangles &c.

Q.E.D

[*Appendix* A. § 11.

PROP. XL. THEOREM.

Equal Triangles, on equal bases in the same Line, and on the same side of it, are between the same Parallels.



Let equal Triangles ABC, DEF, be on equal bases BC, EF, in same Line BF, and on same side of it. It is to be proved that they are between same Parallels.

Join AD.

Now, if AD be not parallel to BF, from A draw AG parallel to BF, and meeting ED, or ED produced, at G; and join GF; and first, let AG fall below AD.

Then, $\because$ Triangles ABC, GEF, are on equal bases and between same Parallels,

$\therefore$ Triangle GEF is equal to Triangle ABC; [Prop. 38.

i. e. to Triangle DEF, the part equal to the whole, which is absurd.

Similarly, if AG fall above AD.

$\therefore$ AD is parallel to BF.

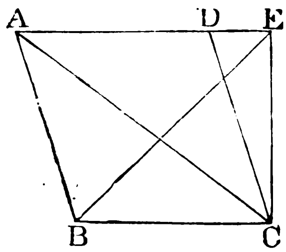
Therefore equal Triangles, &c.

Q.E.D.

[*Appendix* A. § 12.

PROP. XLI. THEOREM.

If a Parallelogram and a Triangle be on the same base and between the same Parallels: the Parallelogram is double of the Triangle.



Let Parallelogram DB and Triangle EBC be on same base BC and between same Parallels AE, BC. It is to be proved that DB is double of EBC.

Join AC.

Then, $\because$ Triangles ABC, EBC, are on same base and between same Parallels,

$\therefore$ they are equal. [Prop. 37.

But Parallelogram DB is double of triangle ABC;

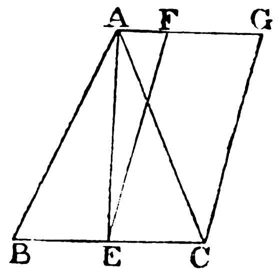
$\therefore$ it is also double of Triangle EBC.

Therefore, if a Parallelogram &c.

Q.E.D.

PROP. XLII. PROBLEM.

To describe a Parallelogram equal to a given Triangle, and having an angle equal to a given angle.



Let ABC be given Triangle, and D given angle. It is required to describe a Parallelogram equal to Triangle ABC, and having an angle equal to angle D.

Bisect BC at E; join AE; at E, in Line EC, make angle CEF equal to given angle D; from A draw AFG parallel to BC; and from C draw CG parallel to EF.

Then FC is a Parallelogram.

∵ Triangles ABE, AEC are on equal bases and between same parallels,

∴ they are equal; [Prop. 38.

∴ Triangle ABC is double of Triangle AEC.

Also, ∵ Parallelogram FC and Triangle AEC are on same base and between same parallels,

∴ Parallelogram FC is double of Triangle AEC; [Prop. 41.

∴ Parallelogram FC is equal to Triangle ABC;

and it has angle AEC equal to given angle D;

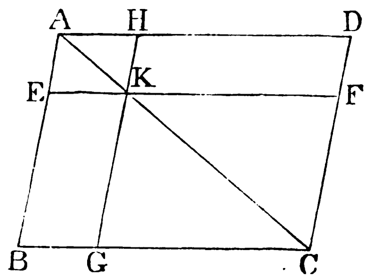
i.e. a Parallelogram has been described equal to the given Triangle, and having an angle equal to the given angle.

Q.E.F.

[*Appendix* A. § 13.

PROP. XLIII. THEOREM.

The complements of a Parallelogram are equal.



Let $ABCD$ be a Parallelogram; let AC be a diagonal; and through K, a Point in AC, draw EF, HG parallel to sides. It is to be proved that complements EG, HF are equal.

$\because$ BD is a Parallelogram and AC its diagonal,

$\therefore$ Triangle ABC is equal to Triangle ADC; [Prop. 34.

similarly Triangles AEK, KGC, parts of the one, are respectively equal to Triangles, AHK, KFC, parts of the other;

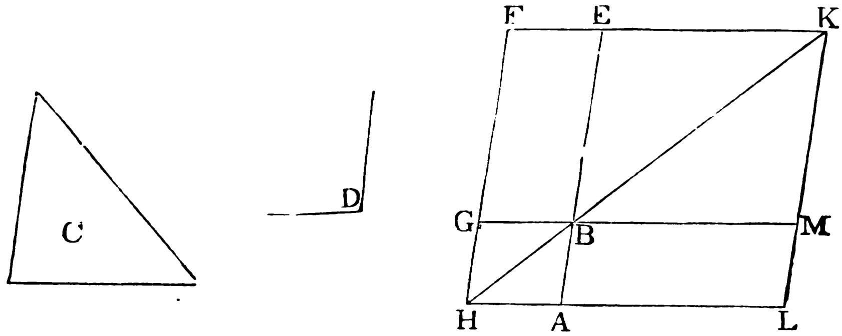
$\therefore$ remainder EG is equal to remainder HF.

Therefore the complements of a Parallelogram are equal.

Q.E.D.

PROP. XLIV. PROBLEM.

On a given line to describe a Parallelogram equal to a given Triangle and having an angle equal to a given angle.



Let AB be given line, C given Triangle, and D given angle. It is required to describe on AB a Parallelogram equal to Triangle C, and having an angle equal to angle D.

Make Parallelogram FB equal to Triangle C, and having angle GBE equal to angle D, so that BE may be in same Line with AB; produce FG to H; from A draw AH parallel to GB or FE; and join HB.

$\because$ FH meets Parallels FE, HA,

$\therefore$ angles EFH, FHA, are together equal to two right angles; [Prop. 29.

$\therefore$ angles EFH, FHB, are together less than two right angles;

$\therefore$ FE, HB will meet if produced. [Ax. § 16.

Produce FE, HB to meet at K; from K draw KL parallel to FH or EA; and produce GB, HA, to M, L.

Then FL is a Parallelogram.

Also, $\because$ FB, BL are its complements,

$\therefore$ BL is equal to FB; [Prop. 43.

i.e. to given Triangle C;

also angle ABM is equal to vertical angle GBE; [Prop. 15.

i.e. to given angle D;

also BL is described on given line AB.

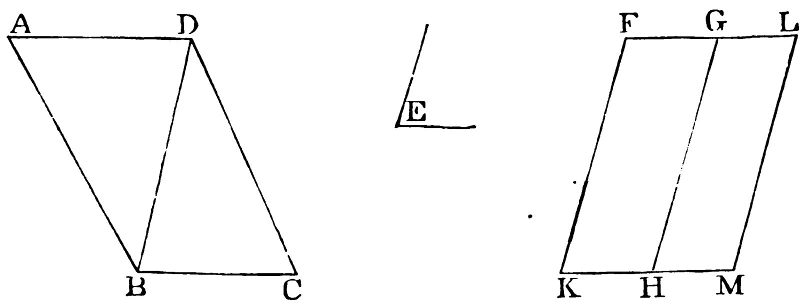
Q.E.F.

[*Appendix* A. § 14.

PROP. XLV. PROBLEM.

To describe a Parallelogram equal to a given rectilinear Figure and having an angle equal to a given angle.

First, let given Figure be quadrilateral.



Let $ABCD$ be given Figure, and E given angle. It is required to describe a Parallelogram equal to $ABCD$, and having an angle equal to angle E.

Join DB; describe Parallelogram FH equal to Triangle ABD and having angle K equal to angle E; [Prop. 42.

on GH describe Parallelogram GM equal to Triangle DBC and having angle GHM equal to angle E. [Prop. 44.

$\because$ KH meets Parallels FK, GH,

$\therefore$ angles K, KHG, are together equal to two right angles; [Prop. 29.

but angle K is equal to angle GHM, each being equal to angle E;

$\therefore$ angles GHM, KHG, are together equal to two right angles;

$\therefore$ KH, HM are in same Line. [Prop. 14.

Again, $\because$ *GH* meets Parallels *FG*, *KM*,

$\therefore$ alternate angles *FGH*, *GHM* are equal. [Prop. 29.

Again, $\because$ *GH* meets Parallels *GL*, *HM*,

$\therefore$ angles *LGH*, *GHM*, are together equal to two right angles.

But angle *GHM* is equal to angle *FGH*;

$\therefore$ angles *LGH*, *FGH*, are together equal to two right angles;

$\therefore$ *FG*, *GL* are in same Line.

Again, $\because$ *FK*, *LM* are parallel to *GH*,

$\therefore$ they are parallel to each other. [Prop. 30.

$\therefore$ *FM* is a Parallelogram.

Also *FM* is equal to Parallelograms *FH*, *GM*,

i.e. to Triangles *ABD*, *DBC*,

i.e. to given Figure *ABCD*;

and it has angle *K* equal to given angle *E*.

Q.E.F.

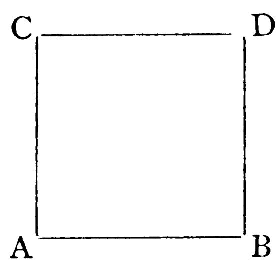
Similarly, if the given Figure have five or more sides.

Corollary.

From this it is manifest how on a given Line to describe a Parallelogram equal to a given rectilinear Figure and having an angle equal to a given angle.

PROP. XLVI. PROBLEM.

On a given Line to describe a Square.



Let AB be given Line. It is required to describe a Square on it.

From A draw AC at right angles to AB, making it equal to AB; from C draw CD parallel to AB; and from B draw BD parallel to AC.

Then AD is a Parallelogram;

$\therefore$ BD is equal to AC, and CD to AB; [Prop. 34.

but AC was made equal to AB;

$\therefore$ AD is equilateral.

Again, $\because$ AC meets Parallels CD, AB,

$\therefore$ angles C, A, are together equal to two right angles;

but angle A is right;

$\therefore$ angle C is also right; [Prop. 13. Cor. 3.

also, $\because AD$ is a Parallelogram,

$\therefore$ angle D is equal to angle A, and angle B to angle C;
[Prop. 34.

$\therefore AD$ is rectangular.

Therefore it is a Square;

and it is described on the given line AB.

Q.E.F.

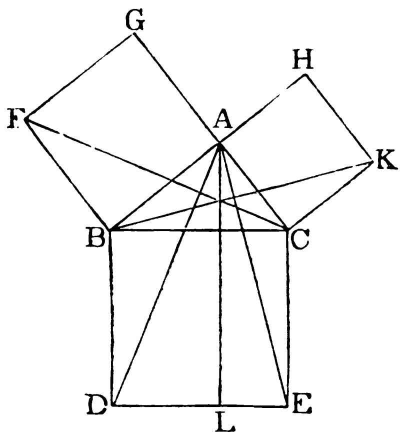
[*Appendix* A. § 15.

Corollary.

If a Parallelogram have one angle right, it is rectangular.

PROP. XLVII. THEOREM.

The square of the hypotenuse of a right-angled Triangle is equal to the squares of the sides.



Let ABC be right-angled Triangle, having angle A right. It is to be proved that square of BC is equal to squares of AB, AC.

On AB, AC, BC, describe Squares AF, AK, BE; through A draw AL parallel to BD or CE; and join AD, CF.

$\because$ angles BAG, BAC, are together equal to two right angles,

$\therefore$ AG, AC are in same Line. [Prop. 14.

Now angles CBD, ABF, are equal, being right;
to each add angle ABC;

$\therefore$ angle ABD is equal to angle FBC;

also FB is equal to BA, being sides of a Square;
similarly BC is equal to BD;

then, in Triangles ABD, FBC,

$\because \begin{cases} AB, BD, \text{ are respectively equal to } FB, BC, \\ \text{and angle } ABD \text{ is equal to angle } FBC, \end{cases}$

$\therefore$ Triangles are equal; [Prop. 4.

but Parallelogram BL is double of Triangle ABD, because they are on same base and between same Parallels; [Prop. 41.

similarly Square AF is double of Triangle FBC;

$\therefore$ Parallelogram BL is equal to Square AF. [Ax. § 6.

In the same way it may be proved that Parallelogram CL is equal to Square AK.

$\therefore$ Square BE is equal to Squares AF, AK; [Ax. § 2.

i. e. square of BC is equal to squares of AB, AC.

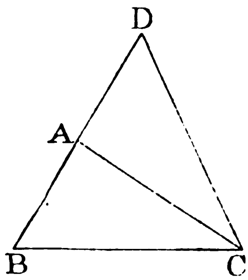
Therefore the square of the &c.

Q.E.D.

[*Appendix* A. § 16.

PROP. XLVIII. THEOREM.

If the square of one side of a Triangle be equal to the squares of the other two sides: the angle contained by these two sides is right.



Let ABC be a Triangle such that square of BC is equal to squares of AB, AC. It is to be proved that angle BAC is right.

From A draw AD at right angles to AC, making it equal to AB; and join DC.

$\because$ AB is equal to AD,

$\therefore$ square of AB is equal to square of AD;

to each add square of AC;

$\therefore$ squares of AB, AC, are together equal to squares of AD, AC;

but square of BC is equal to squares of AB, AC; [Hyp.

and square of DC is equal to squares of AD, AC; [Prop. 47.

$\therefore$ square of BC is equal to square of DC;

$\therefore$ BC is equal to DC;

then, in Triangles ABC, ADC,

$\because$ AB, AC, BC, are respectively equal to AD, AC, DC,

$\therefore$ angle BAC is equal to angle DAC; [Prop. 8.

$\therefore$ it is a right angle.

Therefore, if the square &c. Q.E.D.

BOOK II.

DEFINITIONS.

I.

A Rectangle is said to be "**contained**" by any two adjacent sides of it.

§ 2.

If certain two Lines be given, the phrase "**the rectangle contained by the two Lines,**" or "**the rectangle of the two Lines,**" denotes the magnitude of any Rectangle which has two adjacent sides respectively equal to the two Lines.

§ 3.

If, through any Point in the Diagonal of a Parallelogram, Lines be drawn parallel to the sides, the Figure made up of one of the Parallelograms about the Diagonal together with the two Complements is called a **Gnomon.**

§ 4.

[If, from the ends of a finite Line, perpendiculars be drawn to another Line produced if necessary, the portion of that other Line, intercepted between the perpendiculars, is called the **projection** of the first Line on the other.]

AXIOMS.

§ 1.

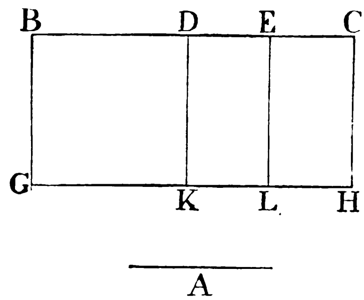
[The rectangle of two equal Lines is equal to the square of one of them.]

§ 2.

[If a finite Line be projected upon a Line parallel to it, its projection is equal to itself: if upon a Line at right angles to it, its projection is a Point.]

PROP. I. THEOREM.

If there be two Lines, of which one is divided into any number of parts: the rectangle of the two Lines is equal to the sum of the rectangles contained by the undivided Line and the several parts of the divided Line.



Let A, BC, be two lines; and let BC be divided at D and E. It is to be proved that rectangle of A, BC, is equal to sum of rectangles of A, BD, of A, DE, and of A, EC.

From B draw BG at right angles to BC, making it equal to A; from G draw GH parallel to BC; and from D, E, C, draw DK, EL, CH, parallel to BG.

Then all these Figures are Rectangles.

Now BH is equal to BK, DL, EH.

But BH is equal to rectangle of A, BC, for BG is equal to A;

similarly BK is equal to rectangle of A, BD;

also DL is equal to rectangle of A, DE, for DK is equal to BG, i.e. is equal to A;

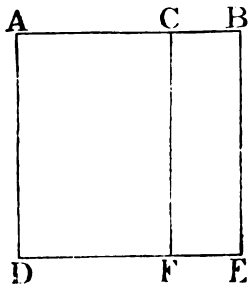
similarly EH is equal to rectangle of A, EC.

Therefore, if there be two Lines, &c.

Q.E.D.

PROP. II. THEOREM.

If a Line be divided into any two parts: the square of the Line is equal to the sum of the rectangles contained by the Line and the two parts.



Let AB be divided at C. It is to be proved that square of AB is equal to sum of rectangles of AB, AC, and of AB, CB.

On AB describe Square $ADEB$; and from C draw CF parallel to AD.

Then AF, CE, are Rectangles.

Now AE is equal to AF, CE.

But AE is equal to square of AB;

also AF is equal to rectangle of AB, AC, for AD is equal to AB;

also CF is equal to rectangle of AB, CB, for BE is equal to AB.

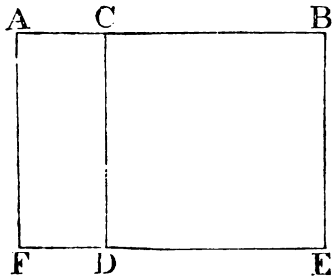
∴ square of AE is equal to sum of rectangles of AB, AC, and of AB, CB.

Therefore, if a Line be divided &c.

Q.E.D.

PROP. III. Theorem.

If a Line be divided into any two parts; the rectangle of the Line and one part is equal to the rectangle of the two parts with the square of the aforesaid part.



Let AB be divided at C. It is to be proved that rectangle of AB, CB is equal to rectangle of AC, CB, with square of CB.

On CB describe Square $CDEB$; produce ED to F; and from A draw AF parallel to CD.

Then AD, AE, are Rectangles.

Now AE is equal to AD, CE.

But AE is equal to rectangle of AB, CB, for BE is equal to CB;

also AD is equal to rectangle of AC, CB, for CD is equal to CB;

and CE is equal to square of CB.

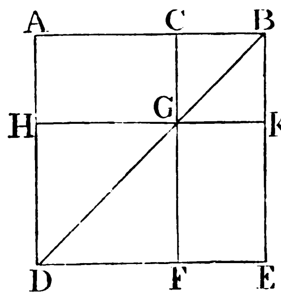
$\therefore$ rectangle of AB, CB, is equal to rectangle of AC, CB, with square of CB.

Therefore, if a Line be divided &c.

Q.E.D.

PROP. IV. THEOREM.

If a Line be divided into any two parts; the square of the Line is equal to the squares of the two parts, with twice their rectangle.



Let AB be divided at C. It is to be proved that square of AB is equal to squares of AC, CB, with twice rectangle of AC, CB.

On AB describe Square $ADEB$; join BD; from C draw CF parallel to AD or BE, cutting BD at G; and through G draw HK parallel to AB or DE.

$\because$ BD cuts Parallels AD, CF,

$\therefore$ exterior angle BGC is equal to interior opposite angle ADB; [I. 29.

also, $\because$ AD is equal to AB,

$\therefore$ angle ADB is equal to angle ABD; [I. 5.

$\therefore$ angle BGC is equal to angle ABD;

$\therefore$ CG is equal to CB;

but BK is equal to CG, and GK to CB; [I. 34.

$\therefore$ CK is equilateral;

also, $\because$ angle CBK is right,

$\therefore$ CK is rectangular; [I. 46. Cor.

$\therefore$ CK is a Square.

Similarly HF is a Square, and is equal to the square of AC, for HG is equal to AC; [I. 34.

Again $\because$ AG, GE are equal, being complements, [I. 43.

$\therefore$ AG, GE, are together equal to twice AG;

i.e. to twice rectangle of AC, CB;

but AE is equal to HF, CK, AG, GE;

that is, square of AB is equal to squares of AC, CB, with twice their rectangle.

Therefore, if a Line be divided &c.

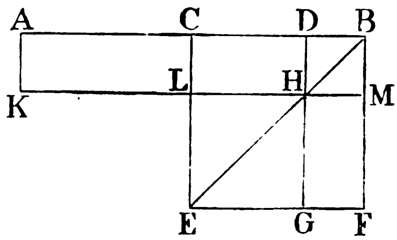
Q.E.D.

Corollary.

Parallelograms about a diagonal of a Square are Squares.

PROP. V. THEOREM.

If a Line be divided into two equal and also into two unequal parts; the rectangle of the unequal parts, with the square of the Line between the points of section, is equal to the square of half the Line.



Let AB be divided into two equal parts at C and into two unequal parts at D. It is to be proved that rectangle of AD, DB, with square of CD, is equal to square of CB.

On CD describe Square $CEFB$; join BE; from D draw DG parallel to CE or BF, cutting BE at H; through H draw KM parallel to AB or EF; and from A draw AK parallel to CL.

Then, $\because$ CH, HF are equal, being complements, [I. 43.
to each add DM;

$\therefore$ CM is equal to DF;

but AL, CM are equal, being on equal bases and between same Parallels; [I. 36.

$\therefore$ AL is equal to DF;

to each add CH;

$\therefore$ AH is equal to gnomon CMG;

to each add LG;

$\therefore$ AH, LG, are together equal to gnomon CMG with LG, i e. to CF;

but AH is equal to rectangle of AD, DB, for DH is equal to DB; [II. 4. Cor.

and LG is equal to square of CD, for LH is equal to CD; [I. 34.

and CF is equal to square of CB;

$\therefore$ rectangle of AD, DB, with square of CD, is equal to square of CB.

Therefore, if a Line be divided &c. Q.E.D.

COROLLARY.

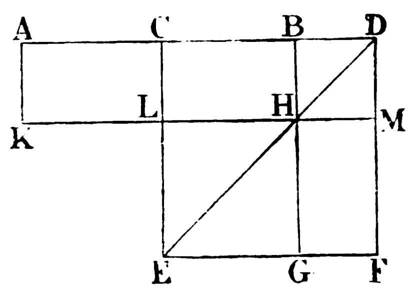
The difference of the squares of two unequal Lines is equal to the rectangle of their sum and difference.

[$\because$ rectangle of AD, DB, with square of CD, is equal to square of CB,

$\therefore$ difference of squares of CB and CD is equal to rectangle of AD, DB; AD being the sum of CB, CD, and DB their difference.]

PROP. VI. THEOREM.

If a Line be bisected and produced to any Point: the rectangle contained by the whole Line thus produced and the produced part, with the square of half the Line, is equal to the square of the Line made up of the half and the produced part.



Let AB be bisected at C and produced to D. It is to be proved that rectangle of AD, DB, with square of CB, is equal to square of CD.

On CD describe Square $CEFD$; join DE; from B draw G parallel to CE or DF, cutting DE at H; through H draw KM parallel to AD or EF; and from A draw AK parallel to CL.

Then, $\because$ $\left\{\begin{array}{l} CH, HF, \text{ are equal, being complements,} \\ \text{and } AL, CH, \text{ are equal, being on equal bases} \\ \quad \text{and between same Parallels,} \end{array}\right.$

$\therefore$ AL is equal to HF;

to each add CM;

$\therefore$ AM is equal to gnomon CMG;

to each add LG;

$\therefore$ AM, LG, are together equal to gnomon CMG with LG,

i.e. to CF;

but AM is equal to rectangle of AD, DB, for DM is equal to DB; [II. 4. Cor.

and LG is equal to square of CB, for LH is equal to CB; [I. 34.

and CF is equal to square of CD;

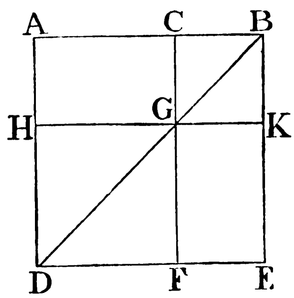
$\therefore$ rectangle of AD, DB, with square of CB, is equal to square of CD.

Therefore, if a Line be bisected &c.

Q.E.D.

PROP. VII. THEOREM.

If a Line be divided into any two parts: the squares of the whole Line and of one part are together equal to twice the rectangle of the whole and that part, with the square of the other part.



Let AB be divided at C. It is to be proved that squares of AB, BC, are together equal to twice rectangle of AB, BC, with square of AC.

On AB describe Square $ADEB$; join BD; from C draw CF parallel to AD or BE, cutting BD in G; and through G draw HK parallel to AB or DE.

Now AG, GE are equal, being complements;

to each add CK;

$\therefore$ AK is equal to CE;

$\therefore$ AK, CE, are together equal to twice AK;

that is, gnomon AKF, with CK, is equal to twice AK;

to each add HF;

$\therefore$ gnomon AKF, with CK and HF, is equal to twice AK, with HF;

but gnomon AKF, with HF, is equal to AE;

$\therefore$ AE, CK, are together equal to twice AK, with HF;

but AE is equal to square of AB;

and CK is equal to square of CB;

and AK is equal to rectangle of AB, BC, for BK is equal to BC; [II. 4. Cor.

and HF is equal to square of AC, for HG is equal to AC; [I. 34.

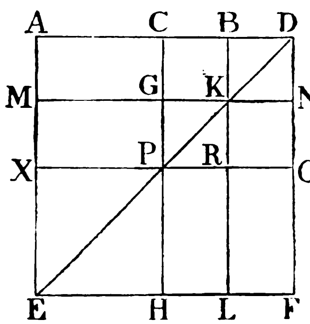
$\therefore$ squares of AB, BC, are together equal to twice rectangle of AB, BC, with square of AC.

Therefore, if a Line be divided &c.

Q.E.D.

PROP. VIII. THEOREM.

If a Line be divided into any two parts: the square of the Line made up of the whole and one part is equal to four times the rectangle of the whole and that part, with the square of the other part.



Let AB be divided at C and produced to D, so that BD is equal to BC. It is to be proved that square of AD is equal to four times rectangle of AB, BC, with square of AC.

On AD describe Square $AEFD$; join DE; from C, B, draw CH, BL, parallel to AE or DF, and cutting DE at P, K; and through P, K, draw $XPRO$, $MGKN$, parallel to AD or EF.

Then XH, GR, BN, are Squares. [II. 4. Cor.

Also, ∵ { CB is equal to BD,
and CB is equal to GK i.e. to GP, [I. 34.
and BD is equal to BK i.e. to CG,

∴ CG is equal to GP;

∴ AG is equal to MP. [I. 36.

Again, $\because \begin{cases} CB \text{ is equal to } BD, \\ \text{and } CB \text{ is equal to } PR, \\ \text{and } BD \text{ is equal to } RO, \end{cases}$ [I. 34.

$\therefore$ PR is equal to RO;

$\therefore$ PL is equal to RF.

But MP, PL are equal, being complements; [I. 43.

$\therefore$ AG, MP, PL, RF, are equal, and together are equal to four times AG.

Again, $\because$ CB is equal to BD,

$\therefore$ CK is equal to BN; [I. 36.

and $\because$ PR is equal to RO,

$\therefore$ GR is equal to KO;

but CK, KO are equal, being complements;

$\therefore$ CK, BN, GR, KO are equal, and together are equal to four times CK;

$\therefore$ gnomon AOH is equal to four times AK;

i. e. to four times rectangle of AB, BC;

to each add XH, which is equal to square of AC, for XP is equal to AC;

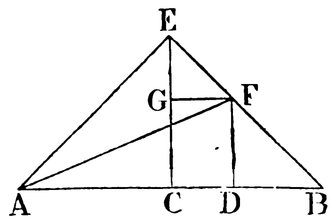
$\therefore$ AF, which is Square on AD, is equal to four times rectangle of AB, BC, with square of AC.

Therefore, if a Line &c.

Q.E.D.

PROP. IX. THEOREM.

If a Line be divided into two equal and also into two unequal parts: the squares of the unequal parts are together twice the squares of half the Line and of the Line between the Points of section.



Let AB be divided into two equal parts at C and into two unequal parts at D. It is to be proved that squares of AD, DB, are together equal to twice squares of AC, CD.

From C draw CE at right angles to AB and equal to AC or CB; join EA, EB; from D draw DF parallel to CE, meeting EB in F; from F draw FG parallel to AB, meeting CE in G; and join AF.

Then, ∵ CA is equal to CE,

∴ angle CEA is equal to angle CAE;

and, ∵ angle ACE is right,

∴ angles CEA, CAE, are together equal to a right angle; [I. 32.

∴ each is half a right angle.

Similarly each of the angles CEB, CBE, is half a right angle;

∴ angle AEB is right.

Again, $\because$ *EC* meets Parallels *GF*, *CD*,

$\therefore$ exterior angle *EGF* is equal to interior opposite angle *ECB*; [I. 29.

$\therefore$ it is a right angle;

also angle *GEF* is half a right angle;

$\therefore$ third angle *GFE* is half a right angle; [I. 32.

$\therefore$ *GF* is equal to *GE*.

Similarly angle *FDB* is right, and *DF* is equal to *DB*.

Again, square of *AE* is equal to squares of *AC*, *CE*; [I. 47.

i. e. to twice square of *AC*.

Similarly square of *EF* is equal to twice square of *GF*;

i.e. to twice square of *CD*; [I. 34.

$\therefore$ squares of *AE*, *EF*, are together equal to twice squares of *AC*, *CD*;

but squares of *AE*, *EF*, are together equal to square of *AF*; i. e. to squares of *AD*, *DF*; i. e. to squares of *AD*, *DB*;

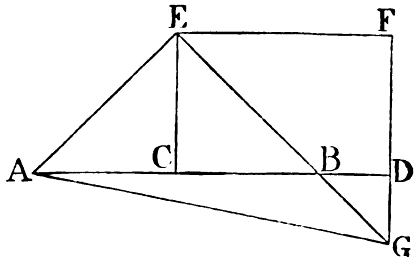
$\therefore$ squares of *AD*, *DB*, are together equal to twice squares of *AC*, *CD*.

Therefore if a Line be divided &c.

Q.E.D.

PROP. X. THEOREM.

If a Line be bisected and produced to any Point: the squares of the whole Line thus produced and of the produced part are together twice the squares of half the Line and of the Line made up of the half and the produced part.



Let AB be bisected at C and produced to D. It is to be proved that squares of AD, DB, are together equal to twice squares of AC, CD.

From C draw CE at right angles to AD and equal to AC or CB; join EA, EB; from D draw DF parallel to CE; and from E draw EF parallel to AD.

∵ EF meets Parallels EC, FD,

∴ angles CEF, EFD, are together equal to two right angles; [I. 29.

∴ angles BEF, EFD, are together less than two right angles;

∴ EB, FD will meet if produced.

Produce them to meet at G; and join AG.

∵ CA is equal to CE,

∴ angle CEA is equal to angle CAE;

and, ∵ angle ACE is right,

∴ angles CEA, CAE, are together equal to a right angle; [I. 32.

∴ each is half a right angle.

Similarly each of the angles CEB, CBE, is half a right angle;

$\therefore$ angle AEB is right.

Again, $\because$ CD meets Parallels EC, FG,

$\therefore$ angle BDG is equal to alternate angle ECB; [I. 29.

$\therefore$ it is a right angle;

also angle DBG is equal to vertical angle EBC; [I. 15.

$\therefore$ it is half a right angle;

$\therefore$ third angle DGB is half a right angle; [I. 32.

$\therefore$ DG is equal to DB.

Again, $\because$ ED is a Parallelogram,

$\therefore$ angle F is equal to opposite angle ECB; [I. 34.

$\therefore$ it is a right angle;

also angle DGB is half a right angle;

$\therefore$ third angle FEG is half a right angle;

$\therefore$ FG is equal to FE.

Again, square of AE is equal to squares of AC, CE; [I. 47.

i.e. to twice square of AC.

Similarly square of EG is equal to twice square of EF;

i.e. to twice square of CD; [I. 34.

$\therefore$ squares of AE, EG, are together equal to twice squares of AC, CD;

but squares of AE, EG are together equal to square of AG; i.e. to squares of AD, DG; i.e. to squares of AD, DB;

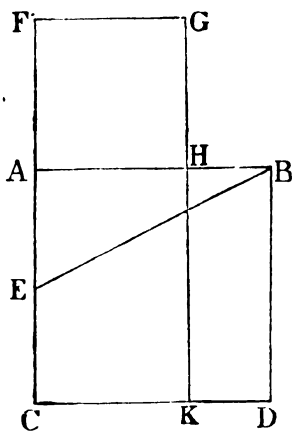
$\therefore$ squares of AD, DB are together equal to twice squares of AC, CD.

Therefore, if a Line be bisected &c.

Q.E.D.

PROP. XI. PROBLEM.

To divide a given Line into two parts, so that the rectangle of the whole and one part may be equal to the square of the other part.



Let AB be the given Line.

On AB describe Square $ACDB$; bisect AC at E; join EB; produce CA to F, making EF equal to EB; on AF describe Square $AFGH$. It is to be proved that rectangle of AB, BH, is equal to square of AH.

Produce GH to K.

Then, $\because$ CA is bisected at E and produced to F,

$\therefore$ rectangle of CF, FA, with square of EA, is equal to square of EF; [II. 6.

i. e. to square of EB; i. e. to squares of EA, AB;

from each take common square of EA;

$\therefore$ rectangle of CF, FA, is equal to square of AB;

i. e. FK is equal to AD;

from each take common part AK;

∴ FH is equal to HD;

but FH is equal to square on AH;

and HD is equal to rectangle of AB, BH, for BD is equal to AB;

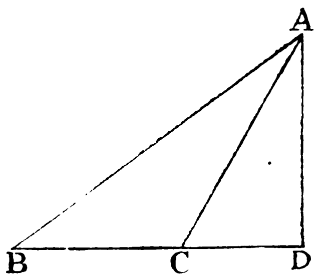
∴ rectangle of AB, BH, is equal to square of AH;

that is, AB has been divided into two parts, so that the rectangle of the whole and one part is equal to the square of the other part.

Q.E.F.

PROP. XII. THEOREM.

In an obtuse-angled Triangle, if a perpendicular be drawn from one of the acute angles to the opposite side produced: the square of the side subtending the obtuse angle is greater than the squares of the sides containing it, by twice the rectangle contained by the side on which, when produced, the perpendicular falls, and the Line intercepted between the perpendicular and the obtuse angle.



Let ABC be a Triangle having the angle ACB obtuse; and from A let AD be drawn at right angles to BC produced. It is to be proved that square of BA is greater than squares of BC, CA, by twice rectangle of BC, CD.

$\because$ BD is divided at C,

$\therefore$ square of BD is equal to squares of BC, CD, with twice rectangle of BC, CD; [II. 4.

to each add square of DA;

$\therefore$ squares of BD, DA, are together equal to squares of BC, CD, DA, with twice rectangle of BC, CD;

that is, square of BA is equal to squares of BC, CA, with twice rectangle of BC, CD;

$\therefore$ square of BA is greater than squares of BC, CA, by twice rectangle of BC, CD.

Therefore, in an obtuse-angled Triangle, &c.

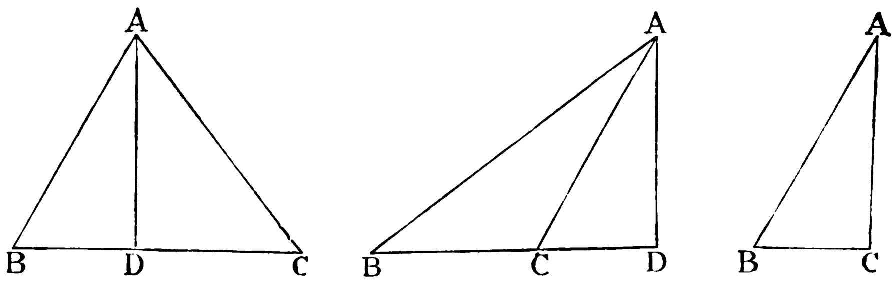
Q.E.D.

[*Note.* The enunciation may also be expressed thus:—

In an obtuse-angled Triangle, the square of the side subtending the obtuse angle is greater than the squares of the sides containing it by twice the rectangle contained by one of them and the projection of the other upon it.]

PROP. XIII. THEOREM.

In any Triangle, if a perpendicular be drawn from one of the angles to the opposite side, produced if necessary: the square of the side subtending another angle, provided it be an acute angle, is less than the squares of the sides containing it, by twice the rectangle contained by the side on which the perpendicular falls, and the Line intercepted between the perpendicular and the acute angle.



Let ABC be a Triangle having the angle B acute.

First, let angle C be acute or obtuse; and from A draw AD at right angles to BC, or BC produced. It is to be proved that square of CA is less than squares of BC, BA by twice rectangle of BC, BD.

In the first figure, $\because$ BC is divided at D, and, in the second figure, $\because$ BD is divided at C,

$\therefore$ in both, squares of BC, BD, are together equal to twice rectangle of BC, BD, with square of CD; [II. 7.

to each add square of DA;

$\therefore$ the three squares of BC, BD, DA, are together equal to twice rectangle of BC, BD, with squares of CD, DA

that is, squares of BC, BA, are together equal to twice rectangle of BC, BD, with square of CA;

$\therefore$ square of CA is less than squares of BC, BA, by twice rectangle of BC, BD.

Next, let angle C be right. It is to be proved that square of CA is less than squares of BC, BA, by twice rectangle of BC, BC.

Now square of BA is equal to squares of BC, CA; [I. 47.

to each add square of BC;

$\therefore$ squares of BC, BA, are together equal to twice square of BC, with square of CA;

$\therefore$ square of CA is less than squares of BC, BA, by twice square of BC, that is, by twice rectangle of BC, BC. [II. Ax. § 1.

Therefore in any Triangle, &c.

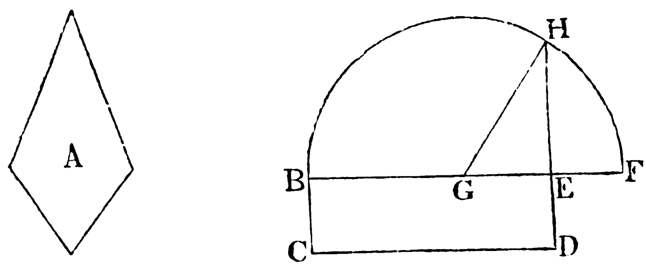
Q.E.D.

[*Note.* The enunciation may also be expressed thus:—

In any Triangle, the square of the side subtending an acute angle is less than the squares of the sides containing it by twice the rectangle contained by one of them and the projection of the other upon it.]

PROP. XIV. PROBLEM.

To describe a Square equal to a given rectilinear Figure.



Let A be the given Figure.

Describe Parallelogram $BCDE$ equal to A, and having one angle right; whence also it is rectangular; [I. 46. Cor.

then, if BE is equal to ED, it is a Square; if not, produce BE to F, making EF equal to ED; bisect BF at G; with centre G, at distance GB or GF, describe Circle BHF; from E draw EH, at right angles to BF, to meet Circle at H; and join GH.

$\because$ BF is divided into two equal parts at G, and into two unequal parts at E,

$\therefore$ rectangle of BE, EF, with square of GE, is equal to square of GF; [II. 6.

i.e. to square of GH; i.e. to squares of GE, EH;

from each take common square of GE;

$\therefore$ rectangle of BE, EF, is equal to square of EH;

but BD is equal to rectangle of BE, EF, for ED is equal to EF;

$\therefore$ BD is equal to square of EH;

but BD is equal to A;

$\therefore$, if a Square be described on EH, it will be equal to given rectilinear Figure A.

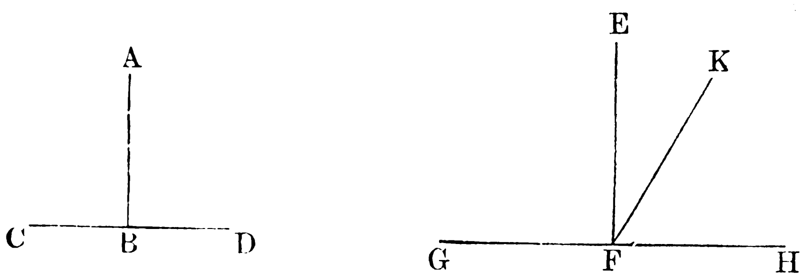
Q.E.F.

APPENDIX A.

NOTES TO BOOK I.

§ 1. AXIOMS, § 14 (3). This, though not formally stated by Euclid, is constantly assumed: e.g. in Prop. 4, where he asserts, '∴ B falls on E,' which would not necessarily follow, if it were possible for AB to fall along DE for a little way, and then to diverge from it.

§ 2. AXIOMS, § 15. This may either be accepted as an Axiom, or proved as a Theorem, thus:—



Let ABC, ABD be equal adjacent angles, and therefore right angles: and similarly for EFG, EFH. It is to be proved that angle ABC = angle EFG.

If the first diagram be applied to the other, so that B may fall on F, and BC along FG;

then BD will fall along FH; [Ax § 14 (3).

then, if BA did not fall along FE, but had another position, as for instance FK, angle KFG would be greater than one of the two equal angles EFG, EFH, and angle KFH would be less than the other;

hence, angle KFG would be greater than angle KFH;

but they are also equal by hypothesis;

which is absurd;

∴ BA will fall along FE;

i.e. angle ABC is equal to angle EFG. Q.E.D.

§ 3. PROP. VII. The fact that a Triangle cannot change its shape without changing the lengths of its sides—a property possessed by no other rectilinear Figure—is of great use in Architecture and the allied Arts. For example, if a gate were made of only *four* pieces of wood, the joints would need great strength to prevent its shape changing from an Oblong to a Rhomboid: but if a *fifth* piece be added as a diagonal, the whole becomes rigid at once, as it now consists of two Triangles.

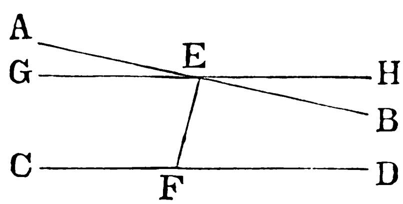
§ 4. PROP. VIII. Euclid might have added, to the Conclusion; 'and the Triangles are equal,' but the words would have been superfluous: for, so soon as the Triangles have been proved to possess the *data* of Prop. IV, the *quaesita* follow naturally.

§ 5. PROP. XVI. Observe that the angles A, B, are differently related to the exterior angle ACD. One may be called its 'interior alternate,' the other its 'interior opposite' angle. The first part of the Proposition proves that the exterior angle is greater than its 'interior alternate' angle. Hence angle BCG is greater than *its* 'interior alternate' angle B.

§ 6. PROP. XXVI. See § 4.

§ 7. PROP. XXVIII, and AXIOMS, § 16. It has long been a matter of dispute whether this Axiom has a real claim to the title, or should be proved as a Theorem. To prove it, however, would need some other Axiom: and no satisfactory substitute for that of Euclid has yet been proposed.

In order to get a clear idea of the subject, it will be well to take the *data* of Euclid's Axiom, and to ascertain what can be *certainly* proved with regard to Lines so situated. Between this, and the conclusion of the Axiom, we shall find a chasm: and, for bridging over this chasm, various methods have been proposed; all, however, involving the assumption of *some* disputable Axiom. (In what follows, the *disputable* matter will be printed in italics.)



Suppose it given that *EF*, meeting *AB* and *CD*, makes the two interior angles *BEF*, *EFD*, together less than two right angles. We desire to reach the conclusion that *AB*, *CD*, will meet if produced.

Let *GEH* be a Line such that the angles *HEF*, *EFD*, are together equal to two right angles.

(1) By Euclid I. 28, we know that *GH* is parallel to *CD*.

(2) It may easily be proved (the student can do this for himself) that the angle *BEF* is less than *HEF*; i.e. that *AB* intersects *GH*, *EB* falling between *GH* and *CD*. Hence,

(3) *AB* intersects a Line which is parallel to *CD*.

We may also assume, as an indisputable Axiom,

(4) Two intersecting Lines cannot both be equidistant from a third Line: i.e. one or other of them must be approaching that third Line. (N.B. A Line is said to 'approach' another, when it passes through two Points, on the same side of the other Line, of which the second is nearer, to that other Line, than the first is.)

This is as far as we can get, on sure ground. The following steps are possibly what Euclid intended should be taken, in order to arrive at his Axiom. The first of them is a disputable Axiom, which he has not laid down in words, but seems to have intended us to assume tacitly. The others follow logically from it, but are, of course, as disputable as the Axiom on which they rest.

(5) *A Line, which approaches another, will meet it if produced.* Hence,

(6) GH *does not approach* CD, *either way, for, if it did, it would meet it; which we know is not the case.* Hence, by (4)

(7) AB *does approach* CD. Hence, by (5)

(8) AB *will meet* CD.

It will be seen, from this, that the proper place for Euclid's Axiom is *after* Prop. 28, as it has no claim whatever to belief until that has been proved: and even then, it requires at least two tacitly assumed Axioms.

§ 8. PROP. XXXII. Cor. 2. This gives us a simple rule for calculating the size (in terms of a right angle) of each angle of

a 'regular' (i.e. equilateral and equiangular) Polygon of n sides. For the n exterior angles are together equal to 4 right angles; hence each is equal to $\frac{4}{n}$ ths of a right angle; hence each interior angle is equal to $\left(2 - \frac{4}{n}\right)$ of a right angle.

§ 9. Prop. XXXVII. Here it is assumed, without proof, that *BE*, and *CF*, will intersect *AD* produced. This may easily be proved by the help of the following easy deduction, from Prop. 29, and Ax. § 26:—

'A Line, which intersects one of two Parallels, will also meet the other.'

§ 10. Prop. XXXVIII. Here it is assumed, without proof, that *BG*, and *FH*, will intersect *AD* produced. See § 9.

§ 11. Prop. XXXIX. Here it is assumed, without proof, that *BD*, or *BD* produced, will meet a Line drawn, through *A*, parallel to *BC*. See § 9.

§ 12. Prop. XL. Here it is assumed, without proof, that *ED*, or *ED* produced, will meet a Line drawn, through *A*, parallel to *BF*. See § 9.

§ 13. Prop. XLII. Here it is assumed, without proof, that *EF* will meet a Line drawn, through *A*, parallel to *BC*; and also that *AF* produced will meet a Line drawn, through *C*, parallel to *EF*. See § 9.

§ 14. Prop. XLIV. Here it is assumed, without proof, that *FG* produced will meet a Line drawn, from *A*, parallel to *GB*; and also that *GB* produced, and *HA* produced, will meet a Line drawn, from *K*, parallel to *FH* or *EA*. See § 9.

§ 15. Prop. XLVI. Here it is assumed, without proof, that *BD* will meet a Line drawn, from *C*, parallel to *AB*. See § 9.

§ 16. Prop. XLVII. The student should be careful to distinguish the phrases 'Square *AF*' and 'square of *AB*.' The first denotes a particular *Figure*, having various properties, e.g. position, shape, and size. The second merely denotes a single *property* of

the said Figure, and is equivalent to the phrase 'size of Square *AF*.' The first-named property of *AF*, viz. its position, is peculiar to it: no other Figure has the same position: but the other properties it has in common with many other Figures. Hence the phrase 'square of *AB*' does not refer to *AF* more than to any other equal Figure: it is simply a *magnitude*.

The student should note also that it is incorrect to call the Square *AF* '*the* Square described on *AB*,' as is often done: for *two* such Squares may be described, one on each side of *AB*. A similar ambiguity may be noted in Prop. 34, where one of the two diagonals is often spoken of as '*the* diagonal.'

NOTES TO BOOK II.

§ 17. PROP. I. Here (as in I. 47) the student should be careful to distinguish between the phrases '*BH*' and 'rectangle of *A*, *BC*.'

APPENDIX B.

Additional Definitions, given in Euclid, but not needed in Books I, II.

§ 1.

A Term, or Boundary, is the extremity of anything.

§ 2.

A Diameter of a Circle is a right Line drawn through the Centre and terminated both ways by the Circumference.

§ 3.

A Semicircle is the Figure contained by a Diameter and the part of the Circumference cut off by it.

§ 4.

The Centre of a Semicircle is the same as that of the Circle.

§ 5.

Multilateral Figures, or **Polygons,** are those which are contained by more than four right Lines.

§ 6.

An Oblong is a quadrilateral Figure which has its angles right, but not all its sides equal.

§ 7.

A Rhombus is a quadrilateral Figure which has all its sides equal, but not its angles right.

§ 8.

A Rhomboid is a quadrilateral Figure which has its opposite sides equal, but not all its sides equal, nor its angles right.

§ 9.

All quadrilateral Figures, which are not Squares, Oblongs, Rhombuses, or Rhomboids, are called **Trapeziums**.

APPENDIX C.

ADDITIONAL DEFINITIONS, NOT GIVEN IN EUCLID.

§ 1.

[**A Radius** of a Circle is a right Line drawn from the Centre and terminated by the Circumference.]

§ 2.

[If, on a given Line, a rectilinear Figure be so drawn that the Line is one of its sides, it is said to be **applied to** the given Line.]

§ 3.

[**A Definition** explains the meaning of a word or phrase.]

§ 4.

[**A Postulate** is an unproved statement, which we are asked to accept, that a certain thing can be done.]

§ 5.

[**An Axiom** is an unproved statement, which we are asked to accept, that a certain thing is true.]

§ 6.

[**A Proposition** is a proved statement, either that a certain thing can be done, or that a certain thing is true. In the first case it is called **a Problem**: in the second, **a Theorem.**]

§ 7.

When a Proposition is so closely connected with another that its proof is contained in it, or may be readily deduced from it, it is sometimes appended to that other Proposition, and is called **a Corollary** of it.

THE END.